A computational method for the identification of plastic zones and residual stress in elastoplastic structures.

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ABSTRACT

Two inverse problems arising in the context of isothermal elastoplasticity with small strains are dealt with in this paper. Both of them use as input data fullfield displacement measurements on a stress-free part of the boundary at various time increments during the loading and unloading steps of a structure. In the first one, the recovery of the plastic strain fields (and then of the plastic zones) during the process of loading is addressed, whereas in the second one the residual stress field after complete unloading is looked for. The computational method derived here is grounded on the minimization of an error in constitutive equations. An illustration of the performance and accuracy of the fields recovery is given for each type of inverse problem on a L-shaped structure.

Keywords: Inverse problem, plasticity, residual stress, identification, computational method.

Introduction

The problem of exploiting (measured) boundary data on a part of a solid (displacement and stress vector fields) in order to extend the mechanical fields within the solid, or to identify missing or unknown boundary conditions is still partially open but potential applications are extremely numerous in mechanical and material sciences and in industry as well. One promising approach dealing with this problem is first to reformulate it within the continuous framework, taking advantage of the fact that the amount and spatial density of information gained allows to consider that the complete displacement field is available on a part of the boundary. And secondly to reformulate it then as a Cauchy problem taking into account the fact that an overspecified data pair is given on a part of the boundary. Cauchy problems belong to the class of inverse problems and are usually ill-posed in the sense of [1].

In this paper, advantage is taken of the information available on a part of the boundary of a structure in order to set up two inverse or identification problems. Both of them use as input data full-field displacement measurements on a stress-free part of the boundary at various time increments during the loading and unloading steps of a structure. In the first one, the recovery of the plastic strain fields (and then of the plastic zones) during the process of loading is addressed, whereas in the second one the residual stress field after complete unloading is looked for. The computational method derived here is grounded on the minimization of an error in constitutive equations, and extends previous methods dedicated to linear and non linear elasticity ([7][5]).

The identification problems

Let be given a regular domain Ω, the boundary of which is decomposed into three non overlapping parts Γ_m, Γ_b, and Γ_u. On Γ_b the stress vector $\mathbf{b}$ is prescribed. Γ_m (the subscript m stands for "measurements") is the part where, thanks to DIC's acquisition for example, both displacements U^m and stress vectors F^m (usually zero for the latest) are available, and make up an overspecified boundary data pair. The remaining part Γ_u of the boundary where not any boundary data is known is generally non connected and can possibly contain internal surfaces such as cracks or boundaries of cavities and inclusions.

$$\partial\Omega = \Gamma_b \cup \Gamma_m \cup \Gamma_u \quad \Gamma_i \cap \Gamma_j = \emptyset \quad i, j = m, b, u \tag{1}$$

If the material is elastoplastic and can be described within the framework of isothermal small strains and Generalized Standard Materials [9], the constitutive equation is written in the following incremental format, when choosing the Euler implicit scheme for the time discretization:

$$\left\{\begin{array}{l} \sigma + \Delta\sigma = \frac{\partial W}{\partial \varepsilon}(\epsilon + \Delta\varepsilon - \varepsilon^p - \Delta\varepsilon^p, \alpha + \Delta\alpha), \\ \sigma + \Delta\sigma \in \partial_{\dot{\varepsilon}^p}\Psi(\Delta\varepsilon^p, \Delta\alpha; \varepsilon^p + \Delta\epsilon^p, \alpha + \Delta\alpha) \\ A + \Delta A = -\frac{\partial W}{\partial \alpha}(\varepsilon + \Delta\varepsilon - \varepsilon^p - \Delta\varepsilon^p, \alpha + \Delta\alpha), \\ A + \Delta A \in \partial_{\dot{\alpha}}\Psi(\Delta\varepsilon^p, \Delta\alpha; \varepsilon^p + \Delta\varepsilon^p, \alpha + \Delta\alpha) \end{array}\right. \tag{2}$$

where $\sigma, \varepsilon, \varepsilon^p, \alpha$ are respectively the Cauchy stress tensor, the strain tensor, the plastic strain, and the supplementary internal variables associated with thermodynamic forces A. The potential W is the free energy and Ψ is the positively 1-homogeneous pseudo-potential of dissipation, both are convex functions. $\partial_x\Psi$ stands for the sub-differential of Ψ with respect to x. For the sake of simplicity we shall drop the arguments of the potentials.The two inverse or identification problems can be formulated as follows :

(IP1) *Provided the data* (U^m, F^m) *on* Γ_m, *and* $\boldsymbol{b}$ *on* Γ_b *at various time instants during the loading and unloading of the solid (*$t \in [0, D]$*) are given, to determine the plastic strain field* $\varepsilon^p(\boldsymbol{x}, t)$ *within the solid along the process.*

(IP2) *Provided the data* (U^m, F^m) *on* Γ_m, *and* $\boldsymbol{b}$ *on* Γ_b *at various time instants during the loading and unloading of the solid (*$t \in [0, D]$*) are given, to determine the residual stress field* $\sigma^{res}(\boldsymbol{x}) = \sigma(\boldsymbol{x}, D)$ *within the solid at the end of the process.*

Reformulation as a Cauchy problem

In order to solve problems (IP1) and (IP2), a formulation can be done to recast them into the framework of a Cauchy problem just by considering the conditions that the time increments $(\Delta u, \Delta\sigma, \Delta\varepsilon^p, \Delta A)$ have to fullfil,namely the the incremental evolution equations within the solid. The Cauchy problem is in looking directly for these increments. Then the plastic strain fields at various time instants and the residual stress field can be simply recovered. The Cauchy Problem for incremental plasticity is the following :

(CP) *Provided the data* (U^m, F^m) *on* Γ_m, *and* $\boldsymbol{b}$ *on* Γ_b *at various time instants during the loading and unloading of the solid (*$t \in [0, D]$*) are given, to determine the incremental fields* $(\Delta u, \Delta\sigma, \Delta\varepsilon^p, \Delta\alpha, \Delta A)$ *fulfilling*

$$\left\{\begin{array}{l} div\ [\sigma + \Delta\sigma] = 0\ ,\quad \varepsilon(u + \Delta u) = [\nabla\,(u + \Delta u)]^{sym} \\ \sigma + \Delta\sigma = \frac{\partial W}{\partial \varepsilon}\ \ ,\ \ A + \Delta A = -\frac{\partial W}{\partial \alpha} \\ \sigma + \Delta\sigma \in \partial_{\dot{\varepsilon}^p}\Psi(\Delta\varepsilon^p, \Delta\alpha)\,,\ A + \Delta A \in \partial_{\dot{\alpha}}\Psi(\Delta\varepsilon^p, \Delta\alpha) \\ \Delta u = \Delta U^m,\ \Delta\sigma.n = \Delta F^m\ on\ \Gamma_m,\quad \Delta\sigma.n = \Delta b\, on\ \Gamma_b \end{array}\right. \tag{3}$$

Cauchy Problems solution is extensively studied in the literature but mainly for linear operators (Lamé operator for linear elasticity, Laplace equation for conductivity problems,Stokes equation for fluids etc.). Existence of solution for non linear Cauchy problem have been studied also by Leitao *et al.* ([2] [3]) by a constructive method using a fixed point algorithm similar to the one designed by Kozlov *et al.* [4]. Here an extension of the variational method previously designed by the authors for linear and nonlinear elasticity is developed for dissipative solids governed by an elastoplastic constitutive relation described in the Generalized Standard Materials format.

A computational method for solving the Cauchy problem in plasticity

The general method for solving this problems relies on two steps. First, two families of auxiliary usual incremental problems $\Delta\mathcal{P}_1$ and $\Delta\mathcal{P}_2$ are defined, each one using one only of the overspecified boundary data on Γ_m and a given normal stress vector field η over $[0, D]$ on Γ_u :

$$\left\{\begin{array}{l} div\,[\sigma + \Delta\sigma_i] = 0\ ,\quad \varepsilon(u + \Delta u_i) = [\nabla\,(u + \Delta u_i)]^{sym} \\ \sigma + \Delta\sigma_i = \frac{\partial W}{\partial \varepsilon}\ \ ,\ \ A + \Delta A_i = -\frac{\partial W}{\partial \alpha} \\ \sigma + \Delta\sigma_i \in \partial_{\dot{\varepsilon}^p}\Psi(\Delta\varepsilon^p_i, \Delta\alpha_i) \\ \Delta\sigma_i.n = \Delta b\, on\ \Gamma_b \end{array}\right. \qquad for\quad i = 1, 2 \tag{4}$$

and respectively for $\Delta\mathcal{P}_1$ and $\Delta\mathcal{P}_2$:

$$(\Delta\mathcal{P}_1)\left\{\begin{array}{l} \Delta u_1 = \Delta U^m\ on\ \Gamma_m \\ \Delta\sigma_1.n = \Delta\eta\ on\ \Gamma_u \end{array}\right. \qquad (\Delta\mathcal{P}_2)\left\{\begin{array}{l} \Delta\sigma_2.n = \Delta F^m\ on\ \Gamma_m \\ \Delta\sigma_2.\,n = \Delta\eta\ on\ \Gamma_u \end{array}\right. \tag{5}$$

If a an incremental surface traction field $\Delta\eta_{opt}$ on Γ_u is such that $\Delta u_1 = \Delta u_2 + \textit{Rigid Body Motion}$, the two problems $\Delta\mathcal{P}_1$ and $\Delta\mathcal{P}_2$ will have the same solution $(\Delta\sigma, \Delta\varepsilon^p, \Delta\alpha)$. Therefore the Cauchy Problem is solved with $(\Delta u_1, \Delta\sigma, \Delta\varepsilon^p, \Delta\alpha, \Delta A)$. A general variational solution method can thus be derived by a second step consisting in building an error function $\mathcal{E}$ between the state increments $\Delta e_1 = (\Delta u_1, \Delta\sigma_1, \Delta\varepsilon_1^p, \Delta A_1)$ and $\Delta e_2 = \Delta u_2, \Delta\sigma_2, \Delta\varepsilon_2^p, \Delta A_2)$ as a functional of $\Delta\eta$ and by minimizing it over all the possible surface traction fields increments defined on Γ_u.

Owing to the general form of the constitutive equation and taking advantage of the convexity of the functions W and Ψ, two errors can be derived with suitable properties ([6][7][8]). They are positive quantities and whenever they vanish then the distance between the two state variable increments vanishes together with the distance of their dual counterparts.

$$\begin{cases} \mathcal{E}_W(\Delta\sigma_1, \Delta\varepsilon_1; \Delta\sigma_2, \Delta\varepsilon_2) = (\Delta\sigma_1 - \Delta\sigma_2) : (\Delta\varepsilon_1^e - \Delta\varepsilon_2^e) - (A_1 - A_2).(\Delta\alpha_1 - \Delta\alpha_2) \\ \mathcal{E}_\Psi(\Delta\sigma_1, \Delta\varepsilon_1^p; \Delta\sigma_2, \Delta\varepsilon_2^p) = (\Delta\sigma_1 - \Delta\sigma_2) : (\Delta\varepsilon_1^p - \Delta\varepsilon_2^p) + (A_1 - A_2).(\Delta\alpha_1 - \Delta\alpha_2) \end{cases} \tag{6}$$

A parametrization enables to put different weights on the errors in stored energy and dissipated one, but outstandingly the value of the parameter, that balances exactly between free energy error and dissipated one, leads to what can be called the Drücker error [**?**]. It involves only the stress and strain tensors, and is then an error in mechanical energy.

$$\mathcal{E} = \frac{1}{2}(\Delta\sigma_1 - \Delta\sigma_2) : (\Delta\epsilon_1 - \Delta\epsilon_2). \tag{7}$$

We can then define the general error functional to be minimized in order to get the solution of the Cauchy problem, and then to design the solution method for (IP1) and (IP2):

$$\Delta\eta_{opt} = ArgMin\left[\mathcal{J}_\chi(\Delta\eta)\right] \quad with \quad \mathcal{J}_\chi(\Delta\eta) = \int_\Omega \mathcal{E}_\chi(\Delta e_1(\Delta\eta)), \Delta e_2(\Delta\eta)))d\Omega \tag{8}$$

The Drücker error can be computed by boundary integration on the whole external surface of the body, thanks to the virtual power principle. This feature has been largely exploited previously to improve the global performance of the solution algorithm for linear Cauchy problems, see [5].

Illustration

The computational method was implemented for both problems on a L-shaped structure submitted to an increasing then a decreasing loading. The overspecified Cauchy data were taken on a part of the right side boundary, whereas the unknown data are located on the top boundary and the left side one. The identification of plastic strain field (IP1) was carried out at various steps of the loading. Figure 1 shows the identified equivalent plastic strain compared to reference values. The identification of the residual stress field (IP2) is carried out at the unloading step. Figure 2 show the identified Von Mises Stress compared to reference one. Let us point out that this result derives from the very good identification of the plastic strain field at the onset of unloading (IP1). Indeed, the residual stress field results directly from the geometric incompatibility of the residual plastic strains within the solid. Because the unloading phase is totally elastic, the residual plastic strain field is exactly the same than at the onset of unloading.

Conclusion

We presented in this paper a computational method for the identification of plastic strains fields and plastic zones during the loading process of a structure, and residual stress field after unolading. The method relies on the solution of a nonlinear Cauchy Problem solved by using a specially designed error in constitutive equation between the solutions of two well-posed problems and minimizing it.

Some improvements have still to be made in the computation of the gradient of the error functional for the case of non twice differentiable potentials for which the general adjoint method can not be directly applied. It is generically the case in elastoplasticity (for the pseudo-potential of dissipation).

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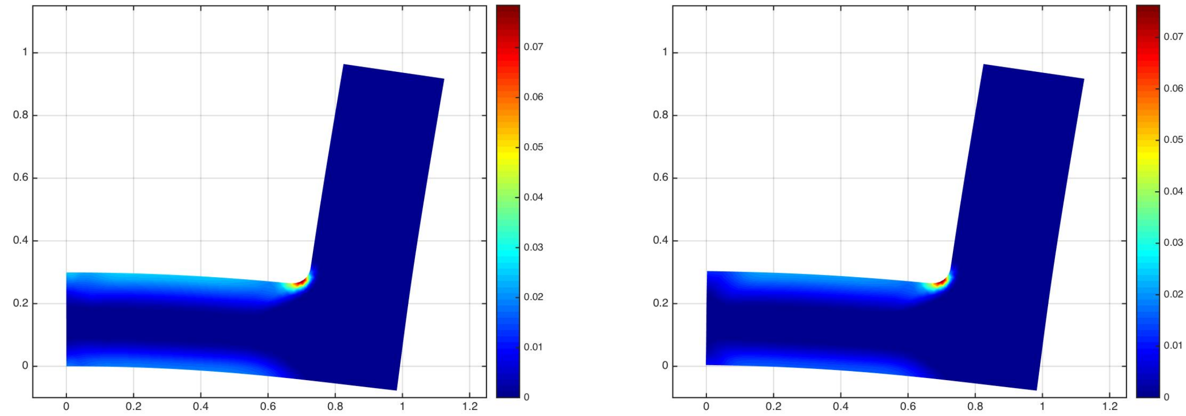
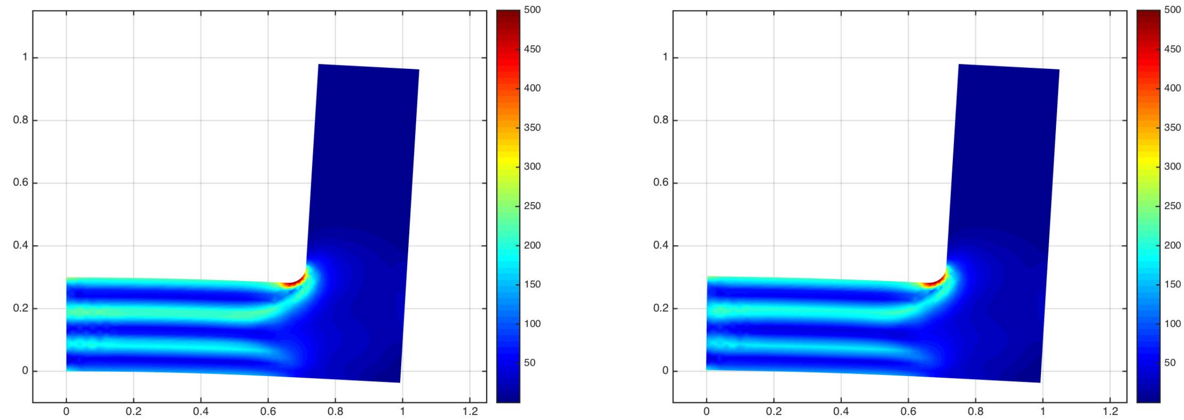


(a) Reference value. (b) Identified value.

Figure 1: Equivalent plastic strain at the end of loading step.



(a) Reference value. (b) Identified value.

Figure 2: Von Mises stress at the end of unloading step.

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