Efficient family of sixth-order iterative methods for nonlinear models which require only one inverse Jacobian matrix

†R. Behl[1], P. Maroju[2] and S.S. Motsa[3]

[1,2,3]Department of Mathematics, Statistics and Computer science, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, Pietermaritzburg, South Africa

†Corresponding author: ramanbehl87@yahoo.in

Abstract

In this study, we design a new efficient families of sixth-order iterative methods for solving scalar as well as system of nonlinear equations. The main beauty of the proposed family is that we have to calculate only one inverse of the Jacobian matrix in the case of nonlinear system which reduce the computational cost. The convergence properties are fully investigated along with two main theorems describing their order of convergence. In addition, we also presented a numerical work which confirm the order of convergence of the proposed family is well deduced for scalar as well as system of nonlinear equations. Further, we have also shown the the implementation of the proposed techniques on real world problems like, Van der Pol equation, Hammerstein integral equation, etc.

Keywords: Nonlinear equations and systems, iterative methods, Newton's method, order of convergence.

Introduction

Construction of higher-order multi-point iterative methods which provide the accurate and efficient approximate solution to the form of

$$F(x) = 0, \tag{1}$$

(where $F : I \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a univariate function when $n = 1$ or multivariate function when $n > 1$ on an open domain I.) is one of the most basic and important problem of the numerical analysis.

The reason behind the importance of this topic is the applicability of these iterative methods in the real world and applied science problems. In the literature, we can find several examples where we can see the applicability of these iterative methods to the real world problems and nonlinear models can be transformed in to the system of nonlinear equations. For example, More presented the set nonlinear model like variational inequalities, the Bratu problem, a shallow arch, etc. in his paper [17]. However, most of them are pharased in the terms of system of nonlinear equations of the form (1). Recently, Rangan et al. [23] discussed the applicability of the nonlinear system on the problem of investigating coarse-grained dynamical properties of neuronal networks in kinetic theory. In addition, Nejat and Ollivier-Gooch [18] presented the problem to study the effect of discretization order on preconditioning and convergence of high-order Newton-Krylor unstructured flow solver in computational fluid dynamics. On the other hand, Grosan and Abraham [11], also shown the applicability of the system of nonlinear equations in neurophysiology, kinematics syntheses problem, chemical equilibrium problem, combustion problem and economics modeling problem. Very recently, Awawdeh [3] and Tsoulos and Stavrakoudis [29], solved the reactor and steering problems by phrasing them in the system of nonlinear equations. Moreover, Lin et al. [16] also discussed the applicability of the system of nonlinear equations in transport theory.

There are two main ways to develop new iterative methods for system of nonlinear equations. Firstly, researchers proposed new iterative methods in order to approximate the zeros of univariate function. Then, they tried to extend the same scheme to the multidimensional case preserving the same order of convergence. For example, Cordero et al. [5], proposed the extension of the classical fourth-order Jarratt's method [13] for scalar equations to system of nonlinear equations. In addition, Abad et al. [1], Cordero et al. [6], Ren et al. [24] and Wang et al. [30], proposed some higher-order extension for systems of nonlinear equations of the previously published work for the scalar equations. Moreover, Sharma and Arora [25] and Hueso et al. [12], also proposed the extension of higher-order Jarratt like method for scalar equation to nonlinear system. We can say that it is one of simple way to develop new scheme for system of nonlinear equations. But, it is not always possible to retain the same order of convergence and the same form of body structure. One of the main reason behind this is that in the case of scalar functional evaluation of the involved function and its derivative consume the same computational cost. However, this is not true in the multidimensional case.

Secondly, researchers tried some other approaches and procedures to develop new and higher-order methods for system of nonlinear equations. In 2010, Sharma et al. [26] proposed fourth and six-order iterative methods based on weighted-Newton iteration. On the other hand, Artidiello et al. [2] proposed fourth-order methods based on the weight function approach. Moreover, Noor et al. [19] also presented several higher-order iterative methods for system of nonlinear with the aid of decomposition technique. We can also use the different approaches like quadrature formulae, Adomian polynomial, divided difference approach, etc. for constructing iterative schemes to solve nonlinear systems. For the details of the other approaches one refers some standard text books [20, 22, 28].

In the earlier proposed schemes by some scholars like Ren et al. [24], Alicia et al. [5], Sharama and Arora [25], Noor et al. [19], Artidiello et al. [2] and Hueso et al. [12], required the evaluation of more than one inverse Jacobian matrix. It is not an easy task to find the inverse of the complicated Jacobian matrix because it requires a lot of computational work. Therefore, we need the higher-order families of iterative methods which require only one evaluation of the Jacobian matrix. Because, it will be very beneficial from the computational point of view.

The principal aim of this study is to propose a new efficient family of sixth-order iterative methods which required only one inverse of the Jacobian matrix for the system of nonlinear equations. Therefore, we propose firstly a new family of sixth-order iterative methods for a scalar equation. Then, we extend this family for the multidimensional case preserving the same order of convergence. The convergence behavior of the proposed methods is tested on a concrete variety of nonlinear equations with same initial guess as other scholars mentioned in their own papers (for the more details please see the section 4). Further, we observed that our proposed methods perform better than the existing ones. Further, we have also shown the applicability of our proposed schemes in the multidimensional case on some real world problems like, Van der pol equation, Hammerstein integral equations and etc.

Development of the scheme for scalar equations

In this section, we propose a new sixth-order family of iterative methods, which is defined as follows:

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)},\\
z_n &= x_n - \left[\theta_1 + \theta_2\frac{f'(y_n)}{f'(x_n)} + \theta_3\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right]\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= z_n - \left[\theta_4 + \theta_5\frac{f'(y_n)}{f'(x_n)} + \theta_6\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right]\frac{f(z_n)}{f'(x_n)},
\end{aligned}
\tag{2}
$$

where $\theta_i \in \mathbb{R},\ i = 1,\ 2,\ \ldots, 6$ are free disposable parameters. The following result demonstrates that the order of convergence reaches sixth-order with some conditions on the disposable parameters.

Theorem 1 *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a sufficiently differentiable function in an interval D containing a simple root α of the equation $f(x) = 0$. Further, we also assume that an initial guess x_0 is sufficiently close to α. Then, the family of iterative methods* (2) *reaches a sixth-order convergence when*

$$\theta_1 = \theta_3 + \frac{7}{4},\ \theta_2 = -2\theta_3 - \frac{3}{4},\ \theta_3 = \frac{9}{8},\ \theta_4 = 1 - \theta_5 - \theta_6,\ \theta_5 = -2\theta_6 - \frac{3}{2}, \tag{3}$$

where $\theta_6 \in \mathbb{R}$, is a free disposable parameter.

Proof. Let us assume that $e_n = x_n - \alpha$ be the error in the n^{th} iteration. Further, let us also expand the functions $f(x_n)$ and it's first order derivative $f'(x_n)$ around the point $x = \alpha$ by using Taylor's series expansion with the assumption $f'(\alpha) \neq 0$, which are defined as follows:

$$f(x_n) = f'(\alpha)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\right), \tag{4}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$ for $k = 2,\ 3,\ \ldots$ and

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^7)\right), \tag{5}$$

respectively.
With the aid of the expressions (4) and (5), we get

$$
\begin{aligned}
\frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 - (4c_2^3 - 7c_3c_2 + 3c_4)e_n^4 + (8c_2^4 - 20c_3c_2^2 + 10c_4c_2 + 6c_3^2\\
&\quad - 4c_5)e_n^5 + \left(52c_3c_2^3 - 16c_2^5 - 28c_4c_2^2 + \left(13c_5 - 33c_3^2\right)c_2 + 17c_3c_4 - 5c_6\right)e_n^6 + O(e_n^7).
\end{aligned}
\tag{6}
$$

By inserting the above expression (6) in the first sub step of scheme (2), we further obtain

$$
\begin{aligned}
y_n - \alpha &= \frac{1}{3}e_n + \frac{1}{3}c_2e_n^2 - \frac{4}{3}(c_2^2 - c_3)e_n^3 + \frac{2}{3}(4c_2^3 - 7c_3c_2 + 3c_4)e_n^4 - \frac{4}{3}\Big(4c_2^4 - 10c_3c_2^2 + 5c_4c_2 + 3c_3^2\\
&\quad - 2c_5\Big)e_n^5 + \frac{2}{3}\left(16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + \left(33c_3^2 - 13c_5\right)c_2 - 17c_3c_4 + 5c_6\right)e_n^6 + O(e_n^7).
\end{aligned}
\tag{7}
$$

Now, we expand the Taylor series expansion of the function $f'(y_n) = f'\left(x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}\right)$ about the point $x = \alpha$ by using (6), which is given as follows:

$$f'(y_n) = f'(\alpha)\left[1 + \frac{2c_2e_n}{3} + \frac{1}{3}\left(4c_2^2 + c_3\right)e_n^2 + \sum_{i=1}^{4} P_i e_n^{i+2} + O(e_n^7)\right], \tag{8}$$

where $P_i = P_i(c_2,\ c_3,\ \ldots,\ c_6)$.
Now, by using the above expressions namely, (4), (5), (6) and (8) in the second sub step, we get

$$z_n - \alpha = (1 - \theta_1 - \theta_2 - \theta_3)e_n + \frac{1}{3}c_2(3\theta_1 + 7\theta_2 + 11\theta_3)e_n^2 + \sum_{l=1}^{4} Q_j e_n^{j+2} + O(e_n^7), \tag{9}$$

where $Q_j = Q_j(\theta_1,\ \theta_2,\ \theta_3,\ c_2,\ c_3,\ \ldots,\ c_6)$.
It is clear from the above equation that for obtaining at least cubic convergence the coefficient of e_n and e_n^2 should be zero simultaneously. Therefore, we have

$$\theta_1 = \theta_3 + \frac{7}{4},\ \ \theta_2 = -2\theta_3 - \frac{3}{4}. \tag{10}$$

Using the above values of θ_1 and θ_2 in $Q_1 = 0$, we obtain the following independent relation

$$8\theta_3 - 9 = 0, \tag{11}$$

which further yields

$$\theta_3 = \frac{9}{8}. \tag{12}$$

By inserting the values of $\theta_1,\ \theta_2$ and θ_3, in the expression (9), we get

$$\begin{aligned} z_n - \alpha = &\left(5c_2^3 - c_3c_2 + \frac{c_4}{9}\right)e_n^4 + \left(-36c_2^4 + 32c_3c_2^2 - \frac{20c_4c_2}{9} - 2c_3^2 + \frac{8c_5}{27}\right)e_n^5 \\ &+ \frac{2}{27}\left(2295c_2^5 - 3537c_3c_2^3 + 633c_4c_2^2 + 9\left(99c_3^2 - 5c_5\right)c_2 - 99c_3c_4 + 7c_6\right)e_n^6 + O(e_n^7). \end{aligned} \tag{13}$$

In this way, we obtain a new optimal fourth-order iterative method. In order to obtain sixth-order convergent family of iterative methods, we expand the Taylor's series expansion of the function $f(z_n)$ about a point $x = \alpha$ with the aid of expression (13), we obtain

$$\begin{aligned} f(z_n) = f'(\alpha)\Bigg[&\left(5c_2^3 - c_3c_2 + \frac{c_4}{9}\right)e_n^4 + \left(-36c_2^4 + 32c_3c_2^2 - \frac{20c_4c_2}{9} - 2c_3^2 + \frac{8c_5}{27}\right)e_n^5 \\ &+ \frac{2}{27}\left(2295c_2^5 - 3537c_3c_2^3 + 633c_4c_2^2 + 9\left(99c_3^2 - 5c_5\right)c_2 - 99c_3c_4 + 7c_6\right)e_n^6 + O(e_n^7)\Bigg]. \end{aligned} \tag{14}$$

By using the equations (4), (5), (8), (13) and (14), in the last sub step of (2), we obtain

$$e_{n+1} = -\frac{1}{9}(45c_2^3 - 9c_3c_2 + c_4)(\theta_4 + \theta_5 + \theta_6 - 1)e_n^4 + \sum_{l=1}^{2} R_l e_n^{l+4} + O(e_n^7), \tag{15}$$

where $R_l = R_l(\theta_4,\ \theta_5,\ \theta_6,\ c_2,\ c_3,\ \ldots,\ c_6)$.
In order to obtain at least fifth-order of convergence, we have to substitute the following value of the disposable parameter θ_4

$$\theta_4 = -\theta_5 - \theta_6 + 1. \tag{16}$$

Now, we will use the above value of θ_4 in $R_1 = 0$, we have

$$2\theta_5 + 4\theta_6 + 3 = 0, \tag{17}$$

which further yields

$$\theta_5 = -2\theta_6 - \frac{3}{2}. \tag{18}$$

By using the values of θ_4 and θ_5 in the expression (15), we get

$$e_{n+1} = -\frac{1}{81}(45c_2^3 - 9c_3c_2 + c_4)\Big(2c_2^2(8\theta_6 - 27) + 9c_3\Big)e_n^6 + O(e_n^7), \quad \theta_6 \in \mathbb{R}. \tag{19}$$

Hence, it is straightforward to say from the above error equation that the proposed scheme (2) reaches the sixth-order convergence. This completes the proof. □

Development of the scheme for multi-dimensional case

The previous scheme (2) for scalar equation can be written for the multi-dimensional case as follows:

$$\begin{aligned}
y^{(n)} &= x^{(n)} - \frac{2}{3}F'(x^{(n)})^{-1}F(x^{(n)}),\\
z^{(n)} &= y^{(n)} - \left[\theta_1 I + \theta_2 F'(x^{(n)})^{-1}F'(y^{(n)}) + \theta_3\left(F'(x^{(n)})^{-1}F'(y^{(n)})\right)^{-2}\right]F'(x^{(n)})^{-1}F(x^{(n)}),\\
x^{(n+1)} &= z^{(n)} - \left[\theta_4 I + \theta_5 F'(x^{(n)})^{-1}F'(y^{(n)}) + \theta_6\left(F'(x^{(n)})^{-1}F'(y^{(n)})\right)^{-2}\right]F'(x^{(n)})^{-1}F(z^{(n)}),
\end{aligned} \tag{20}$$

where I is the identity matrix of order n and $\theta_i,\ i = 1,\ 2,\ \ldots, 6$ are free disposable parameters. With the values of the parameters obtained in Theorem 1 we design a parametric family of sixth-order iterative methods for solving nonlinear systems as shows the following theorem. In the proof of this result we use the tools and procedure introduced in [5].

Theorem 2 *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open neighborhood D of its zero α. Suppose that $F'(x)$ is continuous and nonsingular in α and the initial guess $x^{(0)}$ is close enough to α. Then, the iterative schemes defined by* (20) *have order of convergence six when*

$$\theta_1 = \theta_3 + \frac{7}{4},\ \theta_2 = -2\theta_3 - \frac{3}{4},\ \theta_3 = \frac{9}{8},\ \theta_4 = 1 - \theta_5 - \theta_6,\ \theta_5 = -2\theta_6 - \frac{3}{2},$$

where θ_6 is a free disposable parameter.

Proof. Let us assume that $e^{(n)} = x^{(n)} - \alpha$ be the error in the nth-iteration. Further, by developing $F(x^{(n)})$ in a neighborhood of α, we have

$$F(x^{(n)}) = F'(\alpha)\left[e^{(n)} + C_2(e^{(n)})^2 + C_3(e^{(n)})^3\right] + O((e^{(n)})^4), \tag{21}$$

$C_k = \frac{1}{k!}F'(\alpha)^{-1}F^{(k)}(\alpha)$, $k \geq 2$.
Similarly, we obtain

$$F'(x^{(n)}) = F'(\alpha)\left[I + 2C_2e^{(n)} + 3C_3(e^{(n)})^2 + 4C_4(e^{(n)})^3\right] + O((e^{(n)})^4). \tag{22}$$

By using the above expression (22), we further obtain

$$F'(x^{(n)})^{-1} = \left[I - 2C_2 e^{(n)} + (4C_2^2 - 3C_3)(e^{(n)})^2\right] F'(\alpha)^{-1} + O((e^{(n)})^3), \tag{23}$$

With the help of equation (21) and (23), we have

$$F'(x^{(n)})^{-1}F(x^{(n)}) = e^{(n)} - C_2(e^{(n)})^2 + 2\left(C_2^2 - C_3\right)(e^{(n)})^3 + O((e^{(n)})^4), \tag{24}$$

By using the above expression (24) in the first step of (20), we get

$$y^{(n)} - \alpha = \frac{1}{3}e^{(n)} + \frac{2}{3}C_2(e^{(n)})^2 - \frac{2}{3}(2C_2^2 - 2C_3)(e^{(n)})^3 + O((e^{(n)})^4). \tag{25}$$

With aid of the expression (25), we further obtain

$$F'(y^{(n)}) = F'(\alpha)\left[I + \frac{4}{3}C_2 e^{(n)} + \frac{1}{3}(4C_2^2 + C_3)(e^{(n)})^2\right] + O((e^{(n)})^3) \tag{26}$$

By using the equations (23) and (26), we further yield

$$\begin{aligned} F'(x^{(n)})^{-1}F'(y^{(n)}) = I &- \frac{4C_2}{3}e^{(n)} + \left(4C_2^2 - \frac{8C_3}{3}\right)(e^{(n)})^2 \\ &- \frac{8}{27}(36C_2^3 - 45C_3C_2 + 13C_4)(e^{(n)})^3 + O((e^{(n)})^4). \end{aligned} \tag{27}$$

By using equations (24), (27) and the values of disposable parameters θ_1, θ_2 and θ_3, in the second sub step of the scheme (20), we obtain

$$z^{(n)} - \alpha = A_1(e^{(n)})^4 + A_2(e^{(n)})^5 + O((e^{(n)})^6), \tag{28}$$

where A_1 and A_2 depend on constants C_j.

Now, we want to prove that the proposed scheme will reach sixth-order convergence when we will use the previous values of the disposable parameters (which are mentioned in the previous theorem). For this, we develop $F(z^{(n)})$ in a neighborhood of α

$$F(z^{(n)}) = F'(\alpha)\left[A_1(e^{(n)})^4 + A_2(e^{(n)})^5\right] + O((e^{(n)})^6). \tag{29}$$

With the aid of expressions (23), (24), (27), (29) and the values of disposable parameters θ_4 and θ_5 (which are display in the previous theorem), we have

$$\begin{aligned} &\left[\left(\theta_6 + \frac{5}{2}\right)I + \left(-2\theta_6 - \frac{3}{2}\right)F'(x^{(n)})^{-1}F'(y^{(n)}) + \theta_6\left(F'(x^{(n)})^{-1}F'(y^{(n)})\right)^{-2}\right]F'(x^{(n)})^{-1}F(z^{(n)}) \\ &\qquad = A_1(e^{(n)})^4 + A_2(e^{(n)})^5 + \frac{A_1}{9}\left(2C_2^2(8\theta_6 - 27) + 9C_3\right)(e^{(n)})^6 + O((e^{(n)})^7) \end{aligned} \tag{30}$$

Finally, by using (28) and (29) in the last sub step of the proposed scheme (20), we obtain

$$\begin{aligned} x^{(n+1)} - \alpha &= z^{(n)} - \alpha - \left[A_1(e^{(n)})^4 + A_2(e^{(n)})^5 + \frac{A_1}{9}\left(2C_2^2(8\theta_6 - 27) + 9C_3\right)(e^{(n)})^6 + O((e^{(n)})^7)\right] \\ &= \frac{A_1}{9}\left(2C_2^2(8\theta_6 - 27) + 9C_3\right)(e^{(n)})^6 + O((e^{(n)})^7). \end{aligned} \tag{31}$$

Therefore, (20) is a new family of sixth-order iterative methods. □

Numerical experiments

This section is devoted to verify the convergence behavior and computational efficiency of the proposed family of iterative methods which we have proposed in the earlier sections.

Most of the times, some researchers who want to claim that their methods are superior than other existing methods available in the literature. They consider some well-known or standard or self-made examples and manipulate the initial approximations to claim that their methods are superior than other methods. To halt this practice, we consider six numerical examples; first one is chosen from Guem et al. [7]; second one is chosen from Grau and Díaz-Barrero [8]; third one is chosen from Parhi and Gupta [21], fourth one is chosen from Soleymani [27] and fifth one is consider from Ren et al. [24], with same initial guesses which are mentioned in their papers. Further, we also want to see what will happen if we consider different examples and with different initial guesses, which are not mentioned in their papers. Therefore, we consider one more nonlinear equation from Behl et al. [14]. The details of chosen examples or test functions are available in Table 1. Moreover, the considered test functions with their corresponding zeros and initial guesses are also displayed in the same table.

Now, we employ the new sixth-order scheme (2) $\left(\text{for } \theta_6 = 0,\ \frac{27}{8} \text{ and } \theta_6 = \frac{55}{16}\right)$ denoted by (PM_1), (PM_2) and (PM_3), respectively to see the convergence behavior and effectiveness. We shall compare our methods with a higher-order family of double-Newton methods with a bivariate weighting function that is very recently presented by Guem et al. [7], out of them we choose one of their best method (3.8), called by (GKN). In addition, we consider a sixth-order variants of Ostrowski's method proposed by Grau and Díaz-Barrero [8], out of them we choose expression (4–6), described as (GB). Further, we also compare them with a sixth-order multi-point iterative method (2.7) proposed by Parhi and Gupta [21], called by (PG). Moreover, we will compare them with a sixth-order Jarratt method presented by Soleymani [27], out of which we consider method (10), denoted by (SM). Finally, we also compared our methods with some new sixth-order variants of Jarratt's method designed by Ren et al. [24], out of them we choose method (54) (for $\alpha = \frac{5}{10},\ \beta = \frac{12}{10},\ \gamma = \frac{2}{10},\ \delta = \frac{2}{10}$), described as (RWB).

For better comparisons of our proposed methods, we have displayed the errors between the two consecutive iterations $|x_{n+1} - x_n|$, the estimation of the computational order of convergence $\rho = \frac{\log|(x_n - x_n)/(x_{n-1} - x_{n-2})|}{\log|(x_{n-1} - x_{n-2})/(x_{n-2} - x_{n-3})|}$ or $\frac{\log|(x_n - \alpha)/(x_{n-1} - \alpha)|}{\log|(x_{n-1} - \alpha)/(x_{n-2} - \alpha)|}$ and residual error of the corresponding function $(|f(x_n)|)$, corresponding to each test function in Tables 2 and 3.

Further, we also consider a variety of applied examples to further check the validity of theoretical results for nonlinear system. Therefore, we employ the new sixth-order scheme (20) for $\theta_6 = 0,\ \frac{27}{8}$ and $\theta_6 = \frac{55}{16}$ denoted by $(\widehat{PM_1})$, $(\widehat{PM_2})$ and $(\widehat{PM_3})$, respectively, to verify the performance of these methods on the examples 1–3. We shall compare them with a fourth-order Jarratt's method [5] for system of nonlinear equations, denoted by (JM). In addition, we shall compare them with a method (61) that is recently presented by Ren et al. [24], denoted by $(\overline{RWB})$. Further, we also compared our methods with Ostrowski type methods for solving systems of nonlinear equations designed by Grau et al. [9], out of them we consider methods namely, method (5) and method (7), denoted by (GM_1) and (GM_2), respectively. Moreover, we also compared our methods with sixth-order family of iterative method designed by Cordero et al. [5], out of them we choose method (6), denoted by (CM). Finally, we compare our methods with an efficient Jarratt-like methods presented by Sharma [25], we consider method (13) called by (SA).

In the following Tables 4, 5, 7–10, we have displayed the error between two consecutive error in the iterations $\|x^{(n+1)}-x^{(n)}\|$, the computational order of convergence $\rho = \frac{\log\left[\|x^{(n+1)}-x^{(n)}\|/\|x^{(n)}-x^{(n-1)}\|\right]}{\log\left[\|x^{(n)}-x^{(n-1)}\|/\|x^{(n-1)}-x^{(n-2)}\|\right]}$ and residual error of the corresponding function ($\|F(x^{(n)})\|$).

During the current numerical experiments with programming language Mathematica (Version 9), all computations have been done with multiple precision arithmetic with 1000 digits of mantissa, which minimize round-off errors. Let us remark that, in all tables, $a\ e(\pm b)$ denotes $a \times 10^{(\pm b)}$.

Table 1: Test problems

$f(x)$	$Zeros(\alpha)$	x_0
$f_1(x) = 2\cos(x^2) - \log(1 + 4x^2 - \pi) - \sqrt{2}$; [7]	$\sqrt{\frac{\pi}{4}}$	1
$f_2(x) = \left[1 + (1-\gamma)^4\right] x - (1-\gamma x)^4$ $[\gamma = 5]$; [8]	$0.00361710817890406354076835 1\ldots$	0.05
$f_3(x) = x^2 - e^x - 3x + 2$; [21]	$0.2575302854398607604553673\ldots$	2
$f_4(x) = \tan x$; [27]	0	1.2
$f_5(x) = e^{-x} + \cos x$; [24]	$1.7461395304080124176507 03\ldots$	2
$f_6(x) = x^3 + \sin x + 2x$; [14]	0	1

Table 2: Comparison of $|x_{n+1} - x_n|$ for the functions $f_i(x),\ i = 1,\ 2, \ldots, 6$ among listed methods

f_i	x_0	$\|x_{n+1}-x_n\|$ \ρ	GKN	GB	PG	SM	RWB	PM_1	PM_2	PM_3
f_1	1	$\|x_2-x_1\|$	$1.3e(-4)$	$1.5e(-5)$	$1.2e(-4)$	$1.1e(-5)$	$9.7e(-6)$	$2.4e(-6)$	$2.0e(-5)$	$2.0e(-5)$
		$\|x_3-x_2\|$	$1.1e(-28)$	$3.2e(-28)$	$7.7e(-22)$	$1.9e(-28)$	$5.1e(-29)$	$3.2e(-32)$	$1.1e(-26)$	$1.3e(-26)$
		$\|x_4-x_3\|$	$3.1e(-167)$	$3.7e(-164)$	$6.4e(-125)$	$3.6e(-165)$	$9.9e(-169)$	$2.1e(-187)$	$3.5e(-154)$	$9.4e(-154)$
		ρ	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000
f_2	0.05	$\|x_2-x_1\|$	$8.0e(-9)$	$6.8e(-9)$	$1.6e(-8)$	$5.0e(-9)$	$4.5e(-9)$	$3.0e(-106)$	$7.9e(-10)$	$7.9e(-10)$
		$\|x_3-x_2\|$	$3.5e(-49)$	$9.3e(-50)$	$4.5e(-47)$	$1.8e(-50)$	$8.4e(-51)$	$4.1e(-60)$	$1.6e(-56)$	$1.8e(-56)$
		$\|x_4-x_3\|$	$2.6e(-291)$	$6.3e(-295)$	$2.2e(-278)$	$4.5e(-299)$	$3.3e(-301)$	$2.3e(-359)$	$1.3e(-336)$	$2.2e(-336)$
		ρ	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000
f_3	2	$\|x_2-x_1\|$	$3.5e(-2)$	$9.4e(-2)$	$7.6e(-2)$	$9.8e(-15)$	$3.4e(-1)$	$5.2e(-3)$	$1.4e(-2)$	$1.5e(-2)$
		$\|x_3-x_2\|$	$9.3e(-13)$	$1.4e(-10)$	$9.2e(-11)$	$1.8e(-4)$	$5.4e(-7)$	$3.1e(-19)$	$13e(-15)$	$1.5e(-15)$
		$\|x_4-x_3\|$	$3.2e(-76)$	$1.1e(-63)$	$2.9e(-64)$	$1.4e(-26)$	$9.0e(-42)$	$1.3e(-116)$	$7.5e(-94)$	$2.1e(-93)$
		ρ	6.0005	6.0054	6.0004	5.9147	6.0008	6.0039	5.9979	5.9979
f_4	1.2	$\|x_2-x_1\|$	$2.7e(-1)$	$6.4e(-1)$	$4.4e(-1)$	$3.4e(-1)$	$3.7e(-1)$	$4.2e(-1)$	$3.2e(-1)$	$3.2e(-1)$
		$\|x_3-x_2\|$	$6.0e(-6)$	$3.5e(-3)$	$3.4e(-4)$	$4.8e(-5)$	$7.6e(-5)$	$7.3e(-7)$	$9.7e(-6)$	$9.6e(-6)$
		$\|x_4-x_3\|$	$1.6e(-38)$	$5.0e(-19)$	$4.3e(-26)$	$3.6e(-32)$	$9.1e(-31)$	$6.9e(-45)$	$5.1e(-37)$	$4.6e(-37)$
		ρ	7.0150	7.0180	7.0244	7.0436	7.0359	6.6011	6.9218	6.9244
f_5	2.0	$\|x_2-x_1\|$	$2.9e(-6)$	$1.9e(-6)$	$4.9e(-7)$	$6.1e(-7)$	$1.0e(-6)$	$1.e(-5)$	$8.2e(-7)$	$6.5e(-7)$
		$\|x_3-x_2\|$	$9.2e(-37)$	$1.9e(-37)$	$2.2e(-41)$	$9.2e(-41)$	$3.0e(-39)$	$1.2e(-32)$	$1.3e(-39)$	$2.8e(-40)$
		$\|x_4-x_3\|$	$9.4e(-220)$	$1.4e(-223)$	$2.2e(-247)$	$1.1e(-243)$	$1.9e(-234)$	$2.1e(-194)$	$1.6e(-236)$	$2.1e(-240)$
		ρ	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000
f_6	1	$\|x_2-x_1\|$	$3.7e(-3)$	$1.8e(-3)$	$1.8e(-2)$	$4.3e(-2)$	$1.8e(-2)$	$1.0e(-2)$	$2.2e(-4)$	$4.1e(-4)$
		$\|x_3-x_2\|$	$2.9e(-19)$	$3.1e(-21)$	$3.1e(-14)$	$1.1e(-11)$	$2.4e(-14)$	$4.7e(-16)$	$9.2e(-28)$	$7.9e(-26)$
		$\|x_4-x_3\|$	$5.3e(-132)$	$1.3e(-145)$	$1.3e(-96)$	$1.1e(-78)$	$2.2e(-97)$	$2.0e(-109)$	$2.3e(-191)$	$8.0e(-178)$
		ρ	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000	6.0000

Table 3: Comparison of resiual error $|f(x_n)|$ in among listed methods

f_i	x_0	$\lvert x_{n+1}-x_n\rvert$ \ ρ	GKN	GB	PG	SM	RWB	PM_1	PM_2	PM_3
f_1	1	$\lvert f(x_1)\rvert$	$1.3e(-5)$	$1.4e(-4)$	$1.1e(-3)$	$1.1e(-4)$	$9.3e(-5)$	$2.3e(-5)$	$1.9e(-4)$	$2.0e(-4)$
		$\lvert f(x_2)\rvert$	$1.0e(-27)$	$3.1e(-27)$	$7.4e(-21)$	$1.8e(-27)$	$4.8e(-28)$	$3.1e(-31)$	$1.1e(-25)$	$1.3e(-25)$
		$\lvert f(x_3)\rvert$	$3.0e(-166)$	$3.6e(-163)$	$6.1e(-124)$	$3.4e(-164)$	$9.5e(-168)$	$2.0e(-186)$	$3.3e(-153)$	$9.0e(-153)$
f_2	0.05	$\lvert f(x_1)\rvert$	$2.2e(-6)$	$1.9e(-6)$	$4.5e(-6)$	$1.4e(-6)$	$1.3e(-6)$	$8.4e(-8)$	$2.2e(-7)$	$2.2e(-7)$
		$\lvert f(x_2)\rvert$	$9.7e(-47)$	$2.6e(-47)$	$1.3e(-44)$	$5.1e(-48)$	$2.3e(-48)$	$1.1e(-57)$	$4.5e(-54)$	$4.9e(-54)$
		$\lvert f(x_3)\rvert$	$7.1e(-289)$	$1.7e(-292)$	$6.1e(-276)$	$1.2e(-296)$	$9.1e(-299)$	$6.4e(-357)$	$3.7e(-334)$	$6.2e(-334)$
f_3	2	$\lvert f(x_1)\rvert$	$1.3e(-1)$	$3.5e(-1)$	$2.9e(-1)$	4.2	1.3	$2.0e(-2)$	$5.4e(-2)$	$5.5e(-2)$
		$\lvert f(x_2)\rvert$	$3.5e(-12)$	$5.2e(-10)$	$3.5e(-10)$	$6.8e(-4)$	$2.1e(-6)$	$1.2e(-8)$	$4.9e(-15)$	$5.8e(-15)$
		$\lvert f(x_3)\rvert$	$1.2e(-75)$	$4.3e(-63)$	$1.1e(-63)$	$5.1e(-26)$	$3.4e(-41)$	$4.8e(-116)$	$2.8e(-93)$	$7.8e(-93)$
f_4	1.2	$\lvert f(x_1)\rvert$	$2.7e(-1)$	$7.5e(-1)$	$4.8e(-1)$	$3.6e(-1)$	$3.9e(-1)$	$4.5e(-1)$	$3.3e(-1)$	$3.3e(-1)$
		$\lvert f(x_2)\rvert$	$6.0e(-6)$	$3.5e(-3)$	$3.4e(-4)$	$4.8e(-5)$	$7.6e(-5)$	$7.3e(-7)$	$9.7e(-6)$	$9.6e(-6)$
		$\lvert f(x_3)\rvert$	$1.6e(-38)$	$5.0e(-19)$	$4.3e(-26)$	$3.6e(-32)$	$9.1e(-31)$	$6.9e(-45)$	$5.1e(-37)$	$4.6e(-37)$
f_5	2.0	$\lvert f(x_1)\rvert$	$3.4e(-6)$	$2.3e(-6)$	$5.6e(-7)$	$7.0e(-7)$	$1.2e(-6)$	$1.2e(-5)$	$9.6e(-7)$	$7.5e(-7)$
		$\lvert f(x_2)\rvert$	$1.1e(-36)$	$2.1e(-37)$	$2.6e(-41)$	$1.1e(-40)$	$3.5e(-39)$	$1.3e(-32)$	$1.5e(-39)$	$3.3e(-40)$
		$\lvert f(x_3)\rvert$	$1.1e(-219)$	$1.6e(-223)$	$2.5e(-247)$	$1.3e(-243)$	$2.2e(-234)$	$2.5e(-194)$	$1.9e(-236)$	$2.4e(-240)$
f_6	1	$\lvert f(x_1)\rvert$	$1.1e(-2)$	$5.5e(-3)$	$5.4e(-2)$	$1.3e(-1)$	$5.3e(-2)$	$3.0e(-2)$	$6.5e(-4)$	$1.2e(-4)$
		$\lvert f(x_2)\rvert$	$8.7e(-9)$	$9.4e(-21)$	$9.3e(-14)$	$3.4e(-11)$	$7.3e(-14)$	$1.4e(-35)$	$2.8e(-27)$	$2.8e(-27)$
		$\lvert f(x_3)\rvert$	$1.6e(-131)$	$3.8e(-145)$	$4.0e(-96)$	$3.2e(-78)$	$6.6e(-7)$	$6.1e(-109)$	$7.0e(-191)$	$2.4e(-177)$

Example 1 *Let us consider the Van der Pol equation [4, 19], which is defined as follows:*

$$y'' - \mu(y^2-1)y' + y = 0,\ \mu > 0, \tag{32}$$

which governs the flow of current in a vacuum tube, with the boundary conditions $y(0) = 0,\ y(2) = 1$. *Further, we consider the partition of the given interval* $[0,\ 2]$, *which is given by*

$$x_0 = 0 < x_1 < x_2 < x_3 < \cdots < x_n,\ where\ x_i = x_0 + ih,\ h = \frac{2}{n}.$$

Moreover, we assume that

$$y_0 = y(x_0) = 0,\ y_1 = y(x_1),\ \ldots,\ y_{n-1} = y(x_{n-1}),\ y_n = y(x_n) = 1.$$

If, we discretized the above problem (32) *by using the numerical formula for the first derivative and second derivative, which are given by*

$$y'_k = \frac{y_{k+1} - y_{k-1}}{2h},\ y''_k = \frac{y_{k-1} - 2y_k + y_{k+1}}{2h},\ k = 1,\ 2,\ \ldots,\ n-1,$$

then, we obtain a $(n-1)\times(n-1)$ *system of nonlinear equations*

$$2h^2 x_k - h\mu\left(x_k^2 - 1\right)(x_{k+1} - x_{k-1}) + 2\left(x_{k-1} + x_{k+1} - 2x_k\right) = 0.$$

Let us consider $\mu = \frac{1}{2}$ *and initial approximation* $y_k^{(0)} = \left(\frac{1}{2},\ \frac{1}{2},\ \ldots,\frac{1}{2}\right)$. *In this problem, we consider the value of* $n = 7$ *so that we can obtain a* 6×6 *system of nonlinear equations. The*

Table 4: **(Comparison of $\|x^{(n+1)} - x^{(n)}\|$ among listed methods in the Van der Pol equation)**

$\|x^{(n+1)} - x^{(n)}\|$ \ ρ	JM	$\overline{RWB}$	$GM1$	$GM2$	CM	SA	$\widehat{PM_1}$	$\widehat{PM_2}$	$\widehat{PM_3}$
$\|x^{(2)} - x^{(1)}\|$	$1.4e(-3)$	$1.7e(-5)$	$7.0e(-4)$	$7..7e(-4)$	$1.7e(-5)$	$8.5e(-5)$	$6.4e(-5)$	$1.7e(-5)$	$1.5e(-5)$
$\|x^{(3)} - x^{(2)}\|$	$3.0e(-15)$	$2.2e(-34)$	$4.4e(-17)$	$6.5e(-17)$	$2.2e(-34)$	$1.1e(-29)$	$2.0e(-30)$	$2.2e(-34)$	$1.4e(-34)$
$\|x^{(4)} - x^{(3)}\|$	$1.2e(-61)$	$7.9e(-208)$	$9.3e(-70)$	$4.8e(-69)$	$7.9e(-208)$	$2.8e(-179)$	$6.3e(-183)$	$2.3e(-206)$	$9.3e(-208)$
ρ	3.9826	6.0017	3.9883	3.9874	6.0017	6.0153	5.9768	5.9551	5.9989

Table 5: **(Comparison of residual error $\|F(x^{(n)})\|$ among listed methods in the Van der Pol equation)**

$\|F(x^{(n)})\|$	JM	$\overline{RWB}$	$GM1$	$GM2$	CM	SA	$\widehat{PM_1}$	$\widehat{PM_2}$	$\widehat{PM_3}$
$\|F(x^{1)})\|$	$1.4e(-4)$	$2.0e(-5)$	$5.6e(-4)$	$6.0e(-4)$	$2.0e(-5)$	$8.9e(-5)$	$9.2e(-5)$	$2.3e(-5)$	$2.2e(-5)$
$\|F(x^{(2)})\|$	$5.1e(-15)$	$3.4e(-34)$	$7.0e(-17)$	$1.1e(-16)$	$3.4e(-34)$	$2.8e(-29)$	$6.0e(-30)$	$1.1e(-33)$	$7.2e(-34)$
$\|F(x^{(3)})\|$	$1.5e(-61)$	$1.1e(-207)$	$1.5e(-69)$	$7.8e(-69)$	$1.1e(-207)$	$5.5e(-179)$	$1.2e(-182)$	$3.5e(-206)$	$2.4e(-207)$

solutions of this problem is

$$\alpha = (0.3822666\ldots,\ 0.6911725\ldots,\ 0.9234664\ldots,\ 1.076325\ldots,\ 1.143815\ldots,\ 1.118869\ldots)^t.$$

Example 2 *In this example, we consider one of the famous applied science problem which is known as Hammerstein integral equation (see [20, pp. 19-20] to check the effectiveness and applicability of our proposed methods as compared to the other existing methods, is given as follows:*

$$x(s) = 1 + \frac{1}{5}\int_0^1 F(s,t)x(t)^3 dt$$

where $x \in C[0,1]; s,t \in [0,1]$ and the kernel F is

$$F(s,t) = \begin{cases} (1-s)t, t \le s, \\ s(1-t), s \le t. \end{cases}$$

To transform the above equation into a finite-dimensional problem by using Gauss Legendre quadrature formula given as $\int_0^1 f(t)dt \simeq \sum_{j=1}^{8} w_j f(t_j)$, where the abscissas t_j and the weights w_j are determined for $t = 8$ by Gauss Legendre quadrature formula. Denoting the approximations of $x(t_i)$ by $x_i (i = 1, 2, ..., 8)$, one gets the system of nonlinear equations $5x_i - 5 - \sum_{j=1}^{8} a_{ij}x_j^3 = 0$, where $i = 1, 2, ..., 8$

$$a_{ij} = \begin{cases} w_j t_j (1-t_i), j \le i, \\ w_j t_i (1-t_j), i < j. \end{cases}$$

Where the abscissas t_j and the weights w_j are known and given in following table for $t = 8$. The convergence of the methods towards the root

$$X = (1.00209\ldots, 1.00990\ldots, 1.01972\ldots, 1.02643\ldots, 1.02643\ldots, 1.01972\ldots, 1.00990\ldots, 1.00209\ldots)^t,$$

Table 6: (Abscissas and weights of Gauss Legendre quadrature formula for $t = 8$)

j	t_j	w_j
1	0.019855071751231884158219 57...	0.050614268145188129576265 67...
2	0.101666761293186630204223 03...	0.111190517226687235272178 00...
3	0.237233795041835507091130 47...	0.156853322938943643668981 10...
4	0.408282678752175097530261 93...	0.181341891689180991482575 22...
5	0.591717321247824902469738 07...	0.181341891689180991482575 22...
6	0.762766204958164492908869 52...	0.156853322938943643668981 10...
7	0.898333238706813369795776 96...	0.111190517226687235272178 00...
8	0.980144928248768115841780 43...	0.050614268145188129576265 67...

Table 7: (Comparison of $\|x^{(n+1)} - x^{(n)}\|$ among listed methods in the Hammerstein integral equation)

$\|x^{(n+1)} - x^{(n)}\|$ \ ρ	JM	$\overline{RWB}$	$GM1$	$GM2$	CM	SA	$\widehat{PM_1}$	$\widehat{PM_2}$	$\widehat{PM_3}$
$\|x^{(2)} - x^{(1)}\|$	$1.2e(-4)$	$5.7e(-6)$	$5.7e(-6)$	$5.7e(-6)$	$5.7e(-6)$	$5.7e(-6)$	$5.7e(-6)$	$5.7e(-6)$	$5.7e(-6)$
$\|x^{(3)} - x^{(2)}\|$	$4.3e(-20)$	$6.5e(-38)$	$6.5e(-38)$	$8.7e(-38)$	$6.5e(-38)$	$1.7e(-37)$	$1..7e(-37)$	$7.8e(-38)$	$7.8e(-38)$
$\|x^{(4)} - x^{(3)}\|$	$8.1e(-82)$	$1.6e(-229)$	$1.6e(-229)$	$1.2e(-228)$	$1.6e(-229)$	$1.4e(-226)$	$1.4e(-226)$	$5.5e(-229)$	$5.5e(-229)$
ρ	3.9983	5.9987	5.9987	5.9987	5.9987	5.9989	5.9989	6.0067	5.9989

Table 8: (Comparison of residual error $\|F(x^{(n)})\|$ among listed methods in the Hammerstein integral equation)

$\|F(x^{(n)})\|$	JM	$\overline{RWB}$	$GM1$	$GM2$	CM	SA	$\widehat{PM_1}$	$\widehat{PM_2}$	$\widehat{PM_3}$
$\|F(x^{(1)})\|$	$5.4e(-4)$	$2.7e(-5)$	$2.7e(-5)$	$2.7e(-6)$	$2.7e(-5)$	$2.7e(-5)$	$2.7e(-5)$	$2.7e(-5)$	$2.7e(-5)$
$\|F(x^{(2)})\|$	$2.0e(-19)$	$3.1e(-37)$	$3.1e(-37)$	$4.1e(-37)$	$3.1e(-37)$	$8.0e(-37)$	$8.1e(-37)$	$3.6e(-37)$	$3.6e(-37)$
$\|F(x^{(3)})\|$	$3.8e(-81)$	$7.6e(-229)$	$7.6e(-229)$	$5.7e(-228)$	$7.6e(-229)$	$6.5e(-226)$	$6.5e(-226)$	$2.6e(-228)$	$2.6e(-228)$

is tested in the following Tables 4 and 5 on the basis of the initial guess $\left(-\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2}\right)$.

Example 3 *Let us consider the following nonlinear system of nonlinear equation [10]*

$$f_i(x) = x_i - \cos\left(2x_i - \sum_{j=1}^{4} x_j\right), \tag{33}$$

where $i = 1, 2, 3, 4$. *We choose the initial guess* $x^{(0)} = (1,\ 1,\ 1,\ 1)^t$ *for this problem for obtaining the required solution* $\alpha = (0.5149333\ldots,\ 0.5149333\ldots,\ 0.5149333\ldots,\ 0.5149333\ldots)^t$.

Concluding remarks

The main beauty of the proposed family of iterative methods for the system of nonlinear equations is that we have to calculate only one inverse of the Jacobian matrix (i.e. $F'(x^{(n)})$) in the

Table 9: (Comparison of $\|x^{(n+1)} - x^{(n)}\|$ among listed methods in example (3))

$\|x^{(n+1)} - x^{(n)}\|$ \ ρ	JM	$\overline{RWB}$	$GM1$	$GM2$	CM	SA	$\widehat{PM_1}$	$\widehat{PM_2}$	$\widehat{PM_3}$
$\|x^{(2)} - x^{(1)}\|$	$3.7e(-3)$	$3.6e(-4)$	$3.5e(-4)$	$3.5e(-4)$	$3.6e(-4)$	$3.9e(-4)$	$3.9e(-4)$	$3.6e(-4)$	$3.6e(-4)$
$\|x^{(3)} - x^{(2)}\|$	$4.6e(-12)$	$9.3e(-24)$	$8.3e(-24)$	$1.4e(-23)$	$9.3e(-24)$	$5.5e(-23)$	$5.6e(-23)$	$1.2e(-23)$	$1.2e(-23)$
$\|x^{(4)} - x^{(3)}\|$	$1.2e(-47)$	$2.8e(-141)$	$1.6e(-141)$	$5.4e(-140)$	$2.8e(-141)$	$4.9e(-136)$	$4.9e(-136)$	$2.0e(-140)$	$1.4e(-140)$
ρ	4.0004	6.0000	5.9987	6.0000	6.0000	6.0000	6.0000	6.0000	5.9989

Table 10: (Comparison of residual error $\|F(x^{(n)})\|$ among listed methods in example (3))

$\|F(x^{(n)})\|$	JM	$\overline{RWB}$	$GM1$	$GM2$	CM	SA	$\widehat{PM_1}$	$\widehat{PM_2}$	$\widehat{PM_3}$
$\|F(x^{(1)})\|$	$1.0e(-2)$	$9.7e(-4)$	$9.4e(-4)$	$9.6e(-4)$	$9.7e(-4)$	$1.0e(-3)$	$1.1e(-3)$	$9.8e(-4)$	$9.8e(-4)$
$\|F(x^{(2)})\|$	$1.3e(-11)$	$2.5e(-23)$	$2.3e(-23)$	$3.8e(-23)$	$2.5e(-23)$	$1.5e(-22)$	$1.5e(-22)$	$3.4e(-24)$	$3.2e(-23)$
$\|F(x^{(3)})\|$	$3.2e(-47)$	$7.7e(-141)$	$4.2e(-141)$	$1.5e(-139)$	$7.7e(-141)$	$1.3e(-135)$	$1.3e(-135)$	$5.3e(-140)$	$3.9e(-140)$

case of nonlinear system which reduce the computational cost. The convergence properties are fully investigated along with two main theorems describing their order of convergence. We also tested the order of convergence of our proposed families on a concrete variety of numerical experiments and it is found that the order of convergence of the proposed family is well deduced for scalar as well as system of nonlinear equations. Further, our proposed methods perform better than the existing methods on the mentioned numerical examples even though if we choose the same problems with same initial guesses.

Further, the computational accuracy of the iterative methods dependent on several factors like; body structures of the iterative methods, initial guesses, test functions and the sought zeros. We have shown in the numerical experiments that our proposed iterative methods perform better than the existing ones of the same order. But, these results are not always expected because there is no iterative methods till date which shows best accuracy for every test functions. Further, it is also important to note that the behavior of iterative methods for convergence to the required root is depend on asymptotic error constant c_j, test function $f(x)$ and the required root α.

References

[1] Abad, M.F., Cordero, A. and Torregrosa, J.R. (2014) A family of seventh-order schemes for solving nonlinear systems. *Bull. Math. Soc. Sci. Math. Roum.* **57 (105)(2)**, 133–145.

[2] Artidiello, S., Cordero, A., Torregrosa, J.R. and Vassileva, M.P. (2015) Multidimensional generalization of iterative methods for solving nonlinear problems by means of weight-function procedure. *Appl. Math. Comput.* **268**, 1064–1071.

[3] Awawdeh, F. (2010) On new iterative method for solving systems of nonlinear equations. *Numer. Algor.* **54**, 395–409.

[4] Burden, R.L. and Faires, J.D. (2001) Numerical Analysis. *PWS Publishing Company, Boston*.

[5] Cordero, A., Hueso, J.L. and E. Martínez, Torregrosa, J.R. (2010) A modified NewtonJarratt's composition. *Numer. Algor.* **55**, 87–99.

[6] Cordero, A., Maimó, J.G., Torregrosa, J.R. and Vassileva, M.P. (2014) Solving nonlinear problems by Ostrowski-Chun type parametric families. *J. Math. Chem.* **52**, 430–449.

[7] Geum, Y.H., Kim, Y.I. and Neta, B. (2015) On developing a higher-order family of double-Newton

methods with a bivariate weighting function. *Appl. Math. Comput.* **254**, 277–290.

[8] Grau, M. and Díaz-Barrero, J.L. (2006) An improvement to Ostrowski root-finding method. *Appl. Math. Comput.* **173**, 450–456.

[9] Grau-Sánchez, M., Grau, Á. and Noguera, M.(2011) Ostrowski type methods for solving systems of nonlinear equations. *Appl. Math. Comput.* **218**, 2377–2385.

[10] Grau-Sánchez, M., Grau, Á. and Noguera, M. (2011) Frozen divided difference scheme for solving systems of nonlinear equations. *J. Comput. Appl. Math.* **235**, 1739–1743.

[11] Grosan, C. and Abraham A. (2008) A new approach for solving nonlinear equations systems. *IEEE Trans. Syst. Man Cybernet Part A: Syst. Humans* **38**, 698–714.

[12] Hueso, J.L., Martínez, E. and Teruel, C. (2015) Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems *J. Comput. Appl. Math.* **275**, 412–420.

[13] Jarratt, P. (1966) Some fourth order multipoint iterative methods for solving equations. *Math. Comput.* **20**, 434–437.

[14] Kim, Y.I., Behl, R. and Motsa, S.S. (2016) Higher-order efficient class of Chebyshev-Halley type methods. *Appl. Math. Comput.* **273**, 1148–1159.

[15] Kou, J. and Li, Y. (2007) An improvement of the Jarratt method. *Appl. Math. Comput.* **189**, 1816–1821.

[16] Lin, Y., Bao, L. and Jia, X. (2010) Convergence analysis of a variant of the Newton method for solving nonlinear equations. *Comput. Math. Appl.* **59**, 2121–2127.

[17] Moré, J.J. (1990) A collection of nonlinear model problems. *in:E.L. Allgower, K. Georg(Eds.), Computational Solution of Nonlinear Systems of Equations, Lectures in Applied Mathematics, Amer. Math. Soc. Providence, RI* **26**, 723–762.

[18] Nejat, A. and Ollivier-Gooch, C. (2008) Effect of discretization order on preconditioning and convergence of a high-order unstructured Newton-GMRES solver for the Euler equations. *J. Comput. Phys.* **227(4)**, 2366–2386.

[19] Noor, M.A., Waseem, M. and Noor, K.I. (2015) New iterative technique for solving a system of nonlinear equations. *Appl. Math. Comput.* **271**, 446–466.

[20] Ortega, J.M. and Rheinboldt, W.C. (1970) Iterative solution of nonlinear equations in several variables. *Academic Press, New-York.*

[21] Parhi, S.K. and Gupta, D.K. (2008) A sixth order method for nonlinear equations. *Computational Mechanics* **203**, 50–55.

[22] Petković, M.S., Neta, B., Petković, L.D. and J. Džunić, (2012) Multipoint methods for solving nonlinear equations. *Academic Press.*

[23] Rangan, A.V., Cai, D. and Tao, L. (2007) Numerical methods for solving moment equations in kinetic theory of neuronal network dynamics. *J. Comput. Phys.* **221**, 781–798.

[24] Ren, H., Wu,Q. and Bi, W. (2009) New variants of Jarratt's method with sixth-order convergence. *Numer. Algor.* **52**, 585–603.

[25] Sharma, J.R. and Arora, H. (2014) Efficient Jarratt-like methods for solving systems of nonlinear equations. *Calcolo* **51**, 193–210.

[26] Sharma, J.R., Guna, R.K. and Sharma, R. (2013) An efficient fourth order weighted-Newton method for systems of nonlinear equations. *Numer. Algor.* **2**, 307–323.

[27] Soleymani, F. (2011) Revisit of Jarratt method for solving nonlinear equations. *Numer. Algor.* **57**, 377–388.

[28] Traub, J.F. (1964) Iterative methods for the solution of equations. *Prentice-Hall, Englewood Cliffs.*

[29] Tsoulos, I.G. and Stavrakoudis, A. (2010) On locating all roots of systems of nonlinear equations inside bounded domain using global optimization methods. *Nonlinear Anal. Real World Appl.* **11**, 2465–2471.

[30] Wang, X. and Zhang, T. (2013) A family of Steffensen type methods with seventh-order convergence. *Numer. Algor.* **62**, 429–444.