Kernel-based Collocation Method for Deformable Image Registration Model

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Abstract

In clinical application, radiotherapy is the mainstay of cancer treatment. The deformable image registration (DIR) is a medical imaging device which uses to assess how the tumor changes in size and location. DIR is a very important medical guide in radiotherapy treatment. The satisfactory treatment could be achieved if the exact location and extent of target region are accurately determined, thus the surrounding organs could suffer less impact.

In computational field, the DIR has been studied over 30 years in the discipline of bio-medical science. Due to the complexity of the physical phenomenon of a human body, the development of deformable image models could not be derived uniquely. Many applications did not accomplish the clinically satisfactory level. To improve the treatment flexibility, a high level of accuracy is indispensable in delivering right medical dose to each radiotherapy treatment. Thus the deformable image registration remains to be attractive and important research topics in the clinical field.

In general, the DIR is an optimization method used to measure how the tumor changes in size and location. Several forms of deformable image registration models have been established and solved by different numerical methods. This paper adopts the meshless algorithm using the kernel-based collocation method to solve the concerned model. The application and its formulation to the classical model of DIR algorithms are outlined. The application to real-life case study and the corresponding results will be presented and discussed in our further report.

Keywords: Deformable image registration, Meshless, Kernel-based collocation.

Introduction

One of the classical DIR model by Cachier *et al* [1] is established using the diffusion of spatial transformation; its governing equation is derived from Navier-stokes equation and is given by

$$\mathbf{D} = \frac{(\mathbf{m}-\mathbf{s})\nabla\mathbf{s}}{|\nabla\mathbf{s}|^2 + \xi^2\,|(\mathbf{m}-\mathbf{s})|^2} + \frac{(\mathbf{m}-\mathbf{s})\nabla\mathbf{m}}{|\nabla\mathbf{s}|^2 + \xi^2\,|(\mathbf{m}-\mathbf{s})|^2}, \tag{1}$$

where $\mathbf{m}$ is moving image and $\mathbf{s}$ is the static image matrices, $(\mathbf{m}-\mathbf{s})$ are the differential forces between the moving image and the static image. The normalization factor ξ is added to adjust the force strengths. This attempted to normalize the relations between the moving and static images so as to improve the registration convergence rate and stability.

The function $\mathbf{D} = [u_x, u_y]^T$ is the displacement from the deformed image to the static image. This model generates a mapping between the profile of tumor at the t^{th} treatment and the next profile of tumor at $(t+1)^{th}$ treatment. The volumetric analysis and the positional changes of the tumor can be monitored and then adaptive strategies can be used during the course of treatment.

The present study will use the results from this classical Demons iterative algorithm as a reference guide and be compared with results from the kernel radial collocation algorithm. A real-life deformable image registration from a patient with prostate cancer is used as a reference case study. One of the original CT image is shown in *Figure (1)(a)* and the deformed CT images obtained from a different treatment period are shown in *Figure (1)(b).*

Figure 1(a) Original CT image

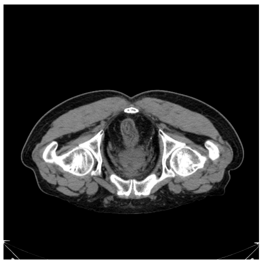

Figure 1(b) Deformed CT image

Kernel-based Collocation Method

This paper focuses on kernel approximations in the form of radial basis functions and apply to solve the differential equation involved in the deformable image model. The method of kernel approximation with radial basis function method have been refined and diversified for facilitating the needs of various types of differential equations. The basic idea of the radial basis interpolation by Hardy [2] is to approximate an unknown displacement function $\{\mathbf{D}(x) : x \in \Omega\}$ by a RBF interpolant at a set of N distinct nodal points $X = \{\mathbf{x}_i \in \Omega : i = 1, 2, \cdots, N\}$.

Let $\Phi : R_+ \to R$ be a set of positive definite radial basis functions defined by

$$\Phi = \{\phi(\|\mathbf{x} - \widetilde{\mathbf{x}}_j\|)\} \quad \mathbf{x}, \widetilde{\mathbf{x}}_j \in \Omega,$$

on a fixed space on Ω. Here ϕ refers to a specific choice of RBF functions that is solely dependent on the Euclidean distance $\| \mathbf{x} - \widetilde{\mathbf{x}}_j \|$ between $\mathbf{x}$ and a fixed centre $\widetilde{\mathbf{x}}_j \in \mathbb{R}^d$. A suitable choice of the function for $\{\mathbf{D}(\mathbf{x}_i) : i = 1, 2, \cdots, N\}$ can ensure the interpolation smoothly passing through the given nodal points in X.

The chosen RBF interpolant for $\mathbf{D}$ can be expressed as a finite linear combination of $\phi(\|\mathbf{x}-\widetilde{\mathbf{x}}_i\|)$ and is given by the equation

$$\mathbf{D}(\mathbf{x}) = \sum_{i=1}^{N} \alpha_i \phi(\|\mathbf{x} - \widetilde{\mathbf{x}}_i\|), \quad \mathbf{x}, \widetilde{\mathbf{x}}_i \in \Omega. \tag{2}$$

The unknown coefficients $\{\alpha_i : i = 1, 2, \cdots, N\}$ can be determined by collocating

$$\widetilde{\mathbf{D}}(\mathbf{x}_i) = \mathbf{D}(\mathbf{x}_i), \text{ for } i = 1, 2, \ldots, N, \tag{3}$$

at a set of N distinct nodal points $\{(x_i, y_i), i = 1, 2, \cdots, N\}$. This yields a system of linear equations which can be expressed in the following matrix form

$$\mathbf{A}_\phi \boldsymbol{\alpha} = \widetilde{\mathbf{D}}, \tag{4}$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ are the unknown coefficients and

$$\widetilde{\mathbf{D}} = [\widetilde{D}(\mathbf{x}_1), \widetilde{D}(\mathbf{x}_2), \ldots, \widetilde{D}(\mathbf{x}_N)]^T.$$

Both $\boldsymbol{\alpha}$ and $\widetilde{\mathbf{D}}$ are $N \times 1$ column matrices, and $\mathbf{A}_\phi = [\phi(x_i - x_j)]_{1 \leq i,j \leq N}$ is an $N \times N$ coefficient matrix.

Generally, the interpolation points in interior and boundaries are distinct and the chosen radial basis function $\phi \in \mathbb{R}^d$ is positive definite, the matrix $\mathbf{A}_\phi$ is always non-singular, so the linear system in (4) has a unique solution by Powell [3]. The unknown coefficients $\boldsymbol{\alpha}$ can then be obtained uniquely by solving the system of linear equations

$$\boldsymbol{\alpha} = \mathbf{A}_\phi^{-1} \widetilde{\mathbf{D}}.$$

The approximated displacement matrix $\mathbf{D}$ can be evaluated once the unknown coefficients $\{\alpha_i, i = 1, \cdots, N\}$ are found.

To prevent the singularity, the radial kernel approximation is formulated by adding a finite number of polynomials into the interpolation system in (2). In the present study, the RBFs interpolant $\mathbf{D}(\mathbf{x})$ in (2) is rewritten as

$$\mathbf{D}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \phi(||\mathbf{x} - \mathbf{x}_j||) + \sum_{k=1}^{M} b_k q_k(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \quad 0 \leq m < N. \tag{5}$$

The terms $\{q_k(\mathbf{x}) : k = 1, 2, \cdots, M\}$ are the radial kernel. In a given set X of distinct nodes $X = \{\mathbf{x}_j \in \Omega : j = 1, 2, \cdots, N\} \subseteq \mathbb{R}^d$, the approximation function in (5) would has a unique solution if the system satisfies the condition

$$\widetilde{\mathbf{D}}(\mathbf{x}_i) = \mathbf{D}(\mathbf{x}_i), i = 1, 2, \cdots, N, \tag{6}$$

and the following constraints

$$\sum_{j=1}^{N} \alpha_j q_k(\mathbf{x}) = 0, \quad k = 1, 2, \cdots, M; \quad j = 1, 2, \cdots, N.$$

The resulting system can be organized in matrix form,

$$\begin{bmatrix} \mathbf{A}_\phi & \mathbf{Q} \\ \mathbf{Q}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}, \tag{7}$$

where $\mathbf{A}_\phi = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ is a square matrix, $\mathbf{a}$ and $\mathbf{c}$ are column vectors. $\mathbf{Q} = [q_k(\mathbf{x}_i)]$ is a $N \times M$ matrix and the unknown coefficients $\mathbf{b}$ is $M \times 1$ matrix given by

$$\mathbf{Q} = \begin{bmatrix} q_1(\mathbf{x}_1) & q_2(\mathbf{x}_1) & \cdots & q_M(\mathbf{x}_1) \\ q_1(\mathbf{x}_2) & q_2(\mathbf{x}_2) & \cdots & q_M(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ q_1(\mathbf{x}_N) & q_2(\mathbf{x}_N) & \cdots & q_M(\mathbf{x}_N) \end{bmatrix}, \quad [\mathbf{b}] = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}.$$

The interpolation problem in (7) is solvable if the matrix of this system is

$$[\widetilde{\mathbf{\Phi}}] = \begin{bmatrix} \mathcal{A}_\phi & \mathbf{Q} \\ \mathbf{Q}^T & 0 \end{bmatrix}$$

is non-singular. In the application of deformable image registration models, the concerned displacement matrix $\mathbf{D}$ can then be determined by the above basis function subject to the given initial values

$$\begin{aligned} \mathbf{D}^0(x, y) &= \mathbf{0}, \\ \mathbf{m}^0(x, y) &= \tilde{\mathbf{m}}^0(x, y), \\ \mathbf{s}^0(x, y) &= \tilde{\mathbf{s}}^0(x, y). \end{aligned}$$

In order to determine the deformable image $\mathbf{m}^j$ at the j^{th} iteration, the forward iterative scheme is applied according to the following equation

$$\mathbf{m}^{*j}(x, y) = \mathbf{m}^{*j-1}(x, y) + \mathbf{D}^{j-1}(x, y)\nabla\mathbf{s}^0. \tag{8}$$

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