Stable node-based smoothed finite element method: application to multiple physical problems

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Abstract

Though the node-based smoothed finite element method (NS-FEM) possesses many superior properties and those prominent inherent properties make it very attractive for researchers, it may be “temporally” unstable when it comes to time-dependent problems. Nevertheless, many physical problems are time-dependent. To apply the NS-FEM to multiple physical problems, effective numerical improvements are essential to cure its temporal instability. Based on this, a stable node-based smoothed finite element method (SNS-FEM) is introduced and the general form of the method is presented in this paper. In the formulation of the SNS-FEM, the simplest linear triangular and tetrahedral elements are employed and the node-based smoothing domain is then constructed on top of element mesh. Gradients variance items of the field variables are considered besides the gradients of the field variables to compute the system stiffness matrix over the smoothing domain. As a result, the system stiffness is strengthened appropriately and the temporal instability of the NS-FEM is able to be cured. The SNS-FEM has been applied to analyze multiple physical problems such as acoustic, heat transfer and electromagnetism problems. Numerical results demonstrate that the SNS-FEM is temporal stability and well suited to various physical problems. In addition to this, the SNS-FEM possesses super accuracy and high computational efficiency.

Keywords: Numerical method, Node-based smoothed finite element method, Temporal stability, Acoustic, Heat transfer, Electromagnetism.

Introduction

The finite element method (FEM) [1]-[3], one of the most powerful numerical methods, is widely used in science and engineering for its simplicity and efficiency. During the past decades, the FEM has been made much progress and extended to almost all areas of engineering and sciences such as structural analysis, mechanical engineering, material science, structure optimization, etc..

The FEM is well suited to various problems with complex geometry. However, in practical applications, the standard finite element method has also been found many limitations and drawbacks [4]-[5] like poor accuracy with lower order elements, sensitive to mesh distortion, volumetric locking phenomenon. These drawbacks seriously restrict the application of the FEM and many numerical improvements [6]-[9] have been developed to solve such issues. Among these methods, the mesh-free method [6],[8],[9] is very promising and achieves many remarkable progresses. As the mesh-free method is implemented beyond the elements, the process is more flexible and the results are insensitive to mesh distortion. Nevertheless, the operations in mesh-free methods are generally more complicated and can be quite costly in terms of the computational effort and resources [10].

In searching for more effective alternatives, Chen and his co-workers proposed a stabilized conforming nodal integration (SCNI) [11],[12] approach based on the strain smoothing techniques. Then, the strain smoothing techniques has been successfully applied to the finite element method by Liu et al. [10],[13]-[19]. And based on this, a series of smoothed finite element methods have been developed such as the cell-based smoothed finite element method (CS-FEM) [13]-[14],[20]-[21], the node-based smoothed finite element method (NS-FEM) [22]-[26], the edge-based smoothed finite element method (ES-FEM) [27-32], the face-based smoothed finite element method (FS-FEM) [33]-[36] and so on [37]-[41]. In recent years, the S-FEMs have been developed greatly and it has been proven that the S-FEMs carry many key features of the standard FEM and mesh-free methods. The S-FEMs is regarded as one of the most promising methods for science and engineering and has already been applied to solve various practical problems [20]-[41].

Among these remarkable S-FEMs, the NS-FEM is regarded as one of the “start” elements by many researchers for it possesses many excellent features in application to solid mechanics problems. The NS-FEM is proposed based on the node-based smoothing technique and the system stiffness matrix is calculated based on the smoothing domain associated with nodes. Studies have shown that the NS-FEM is well immune from the volumetric locking and possesses the upper bound property in strain energy [22]-[23], [25]. In the formulation of NS-FEM, the stress at nodes can be computed directly from the displacement results without any post-processing and it can achieve super-accurate and super-convergent properties of stress solution using the simplest linear triangular and tetrahedral elements. In addition, as the field gradients are obtained directly through the shape functions, i.e., no coordinate transformations are involved, the NS-FEM performs well even severely distorted elements are employed. These dramatic properties make the NS-FEM very attractive for researchers and engineers.

Though the node-based smoothed finite element method (NS-FEM) possesses many superior properties, it has been found temporal instability when it comes to time-dependent problems. It has been proved that the temporal instability is mainly caused by the “overly-soft” property of the NS-FEM model [37]-[38]. To cure the temporal instability of the NS-FEM as well as expand its application limits, various numerical improvements [37],[38],[42]-[50] have been proposed in recent years. Overall, these numerical treatments can be classified into two categories. Beissel and Belytschko [43] added a squared-residual of the equilibrium equation to the potential energy functional as a stabilization term to improve the performance of the nodal integration of the element free Galerkin method where the temporal instability of nodal integration is firstly found. Then the squared-residual stabilization technique has been further developed by Zhang et al. [44], Feng et al. [45] and Wang et al.[46]. These studies can be regarded as the first class of the numerical improvement in which a parameter α is involved in the stabilization process to adjust the system’s stiffness. Another class of numerical improvements is formulated by combining the “overly-soft” NS-FEM with the “overly-stiff” FEM, such as the hybrid smoothed finite element method (HS-FEM) [47]-[50]and the alpha finite element method (α-FEM) [37]-[38]. These methods can also provide very accurate numerical solution with a proper parameter α. However, no matter which class of the method is employed, there is always a parameter which has a great influence on the numerical results. And it’s still an unsolved problem about how to obtain an optimal parameter as both the nature of the problem and the size of mesh discretization will have great influence on the parameter.

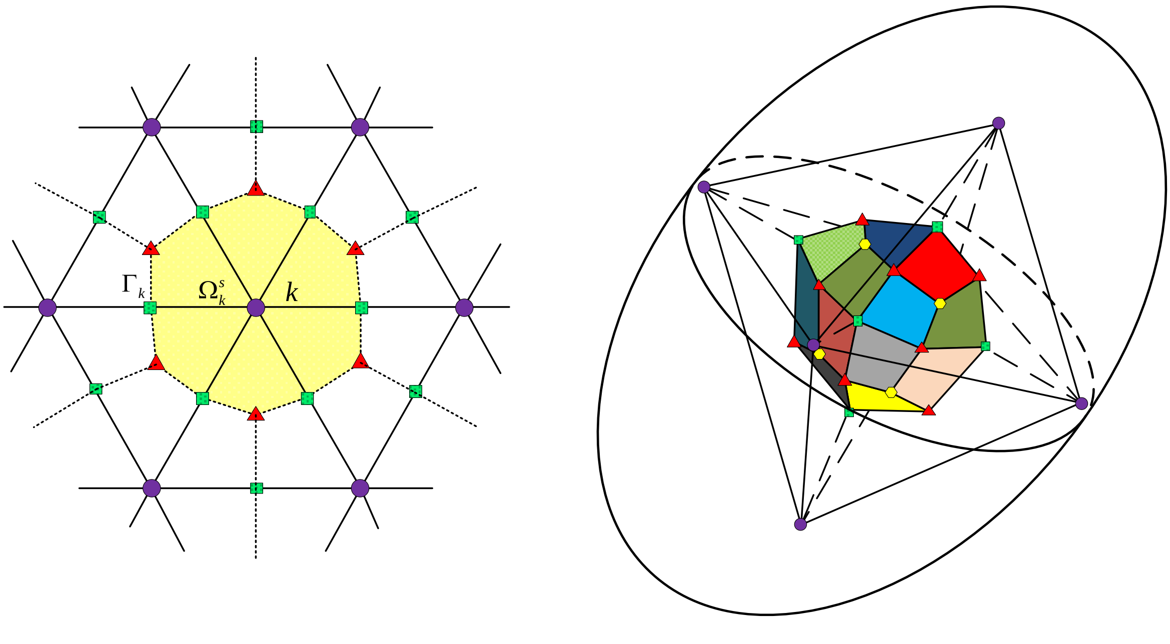
Recently, a stable node-based smoothed finite element method (SNS-FEM) without any uncertain parameter has been proposed by Feng et al. [51] and Wang et al. [52]. In this novel SNS-FEM, the simplest linear triangular and tetrahedral elements are employed to discretize the problem domain and the node-based smoothing domain is then further constructed. Unlike the original NS-FEM, the gradient variances of the field variables are taken into account to construct the stable items to strengthen the system stiffness. As there is no uncertain parameter is introduced in this process, this is a fantastic step forward. The SNS-FEM has been successfully applied to solid mechanics [51], acoustic [52]-[53], heat transfer [54], electromagnetism [55]-[57] and stochastic problems [58]. It's found that the SNS-FEM can be easily extended to multiple physical problems and can generally provide very accurate numerical solutions. Therefore, a general form of the SNS-FEM for multiple physical problems is introduced in this article and its application on acoustic, heat transfer and electromagnetism problems are presented as examples to investigate the performance of the SNS-FEM.

In this work, the application of the SNS-FEM on multiple physical problems will be presented. The rest of the paper is organized as follows: The general form of the SNS-FEM for multiple physical problems is briefly in the next section. After that, numerical examples of different physical problems are studied. Some concluding remarks are made in the last.

General Form of the SNS-FEM

The NS-FEM is one of the "start" elements in the S-FEM family and possesses many excellent features. Based on the node-based strain smoothing technique, the node-based smoothed finite element method is proposed for solid mechanics problems. When extending the NS-FEM to various physical problems, the smoothing operation is carried out based on the gradient of the physical field variables (such as the acoustic pressure, temperature and magnetic vector potential and so on).

Node-based gradient smoothing operation



(a) 2D (b) 3D

● Field node ▲ Centroid of the triangle ■ Mid-edge-point ⬠ Centroid of the tetrahedron

Figure 1. A common process to construct the node-based smoothing domain

In the formulation of the NS-FEM, the numerical integration procedures for the system stiffness matrix are performed based on the node-based smoothing domain which is constructed based on the elements but beyond the elements. Typically, the problem domain Ω is first discretized using N_e triangular or tetrahedral elements in the same manner as in the standard FEM. Based on the obtained background mesh, the problem domain is further subdivided into N_n non-overlapping and non-gap smoothing domains such that $\Omega = \bigcup_{i=1}^{N_n} \Omega_i^s$ and $\Omega_i^s \cap \Omega_j^s = \varnothing \ (i \neq j)$, in which N_n denote the total number of field nodes. Fig. 1 shows a common process to construct the node-based smoothing domain. For an interior node k, in two dimension spaces, the node-based smoothing domain Ω_k^s is constructed by linking the mid-edge-points and the central points of the elements associated with the node k in order. When it comes to three dimension spaces, the smoothing domain Ω_k^s for node k can be constructed by linking the mid-edge-points, the centroids of surface triangles together with the central points of the tetrahedrons associated with the node in proper order.

When the standard FEM is employed to solve some common physical field problems, the field variables within each element can be obtained using the interpolation form

$$\boldsymbol{\varphi} = \sum_{i=1}^{N_p} N_i(\mathrm{x}) \boldsymbol{\varphi}_i \tag{1}$$

where $\boldsymbol{\varphi}$ denotes the physical field variables (scalar: acoustic pressure, temperature; vector: magnetic vector potential); N_p is the number of nodes in each element; $N_i(\mathbf{x})$ is the shape function value at i-th node; $\boldsymbol{\varphi}_i$ is unknown nodal field variables value. Based on the standard Galerkin weak form, the system stiffness matrix $\mathbf{K}$ can be written as the following general form

$$\mathbf{K} = \int_{\Omega} (\nabla \mathbf{N})^{\mathrm{T}} \mathbf{D} (\nabla \mathbf{N}) \mathrm{d}\Omega = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \mathrm{d}\Omega = \sum_{i=1}^{N_e} \int_{\Omega^e} \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \mathrm{d}\Omega \tag{2}$$

in which $\mathbf{B}$ is the general gradient matrix; $\mathbf{D}$ is a matrix of material constants.

Using the gradient smoothing technique and introducing the Green's divergence theorem, the gradient of the physical field variables in the node-based smoothing domain Ω_k^s can be generally expressed as

$$\nabla \overline{\boldsymbol{\varphi}} = \frac{1}{A_k} \int_{\Omega_k^s} \nabla \boldsymbol{\varphi} d\Omega = \frac{1}{A_k} \int_{\Gamma_k^s} \boldsymbol{\varphi} \cdot n d\Gamma = \sum_{i \in M_k} \overline{\mathbf{B}}_i \boldsymbol{\varphi}_i \tag{3}$$

where $\nabla \overline{\boldsymbol{\varphi}}$ represents the smoothed gradient of the field variables; A_k denotes the area or the volume of the smoothing domain; Γ_k^s is the boundary of the node-based smoothing domain; M_k denotes a set containing all nodes located in the influence domain of node k. It should be noted here that Eq. (3) is just a general form of the gradient items and the specific form of a certain physical problem may be slightly different from this. As the Green's divergence theorem is introduced, the area integration over the smoothing domain is converted into the line integral along Γ_k^s. And then, the components of the smoothed gradient matrix $\overline{\mathbf{B}}$ can be written as

$$b_{ip} = \frac{1}{A_k} \int_{\Gamma_k^s} N_i n_p \mathrm{d}\Gamma \quad p = x, y, z \tag{4}$$

Replacing the compatible gradient component shown in Eq. (1) with the smoothed gradient, the smoothed system stiffness matrix can be further explained as

$$\overline{\mathbf{K}} = \sum_{k=1}^{N_n} \overline{\mathbf{K}}_k = \int_{\Omega_k^s} (\overline{\mathbf{B}}_k)^{\mathrm{T}} \cdot (\overline{\mathbf{B}}_k) \mathrm{d}\Omega = (\overline{\mathbf{B}}_k)^{\mathrm{T}} \cdot (\overline{\mathbf{B}}_k) A_k \tag{5}$$

Stabilization of the NS-FEM

To cure the instability of the traditional node-based smoothed finite element method, a stabilization item is constructed and employed to strengthen the system stiffness matrix shown in Eq. (5). As described above, the gradient of the field variables in each smoothing domain is constant, that is, gradient changes over the smoothing domain are ignored. Thus, gradient variances of the field variables are taken into account to construct the stable items. Generally, the smoothing domain is a polygon (2D) or polyhedron (3D) that can be approximated as a circle or sphere domain Ω_k^{sc} with the same area or volume. The equivalent circle or sphere domain is then further divided into four or six sub-domains equally. And based on the obtained sub-domains, four or six points located in x-axis, y-axis, z-axis with the same distance r_e to node k are chosen to be the integration points g_i (i=1,2,…,6), as shown in Fig. 2. The equivalent radius of the approximate domain can be obtained by

$$r_e = \begin{cases} \sqrt{A_k / \pi} & \text{2D} \\ \sqrt[3]{3A_k / 4\pi} & \text{3D} \end{cases} \tag{6}$$

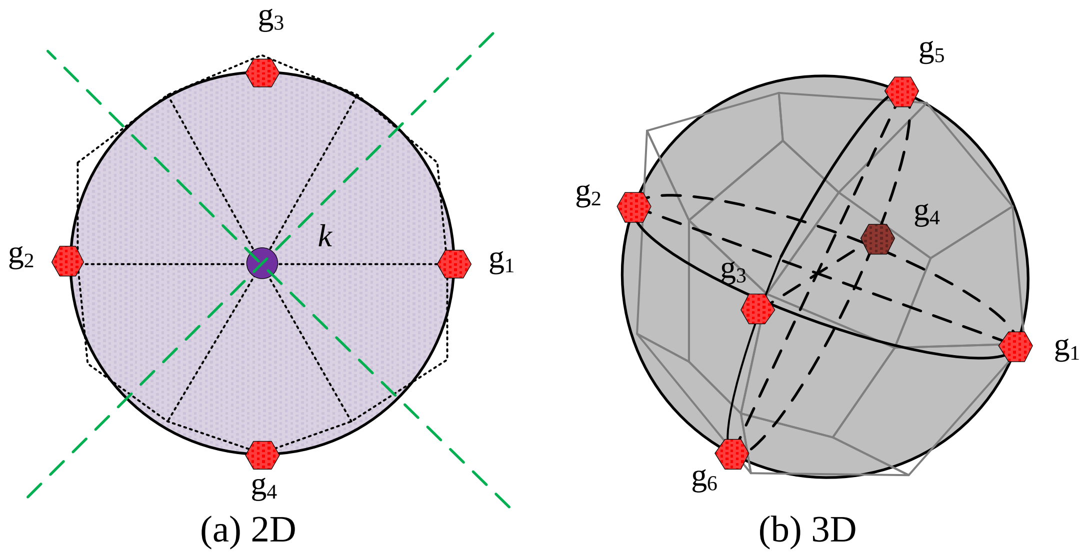


Figure 2. The approximate integration domain and integration points for SNS-FEM

Assuming that the gradient of the field variables in the smoothing domain is continuous and derivable at the first order, thus, the Taylor expansion of the gradient at node k can be expressed as

$$\nabla\boldsymbol{\varphi} = (\nabla\boldsymbol{\varphi})_k + \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial x}(x - x_k) + \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial y}(y - y_k) + \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial z}(z - z_k) \tag{7}$$

After this, the gradient items at each integration points $(\nabla\boldsymbol{\varphi})_{ki}^{sc}$ (i=1, 2, 3, 4, 5, 6) can be obtained and expressed as

$$\begin{aligned}
(\nabla\boldsymbol{\varphi})_{k1}^{sc} &= (\nabla\boldsymbol{\varphi})_k^s + \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial x} r_e \quad (\nabla\boldsymbol{\varphi})_{k2}^{sc} = (\nabla\boldsymbol{\varphi})_k^s - \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial x} r_e \\
(\nabla\boldsymbol{\varphi})_{k3}^{sc} &= (\nabla\boldsymbol{\varphi})_k^s + \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial y} r_e \quad (\nabla\boldsymbol{\varphi})_{k4}^{sc} = (\nabla\boldsymbol{\varphi})_k^s - \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial y} r_e \\
(\nabla\boldsymbol{\varphi})_{k5}^{sc} &= (\nabla\boldsymbol{\varphi})_k^s + \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial z} r_e \quad (\nabla\boldsymbol{\varphi})_{k6}^{sc} = (\nabla\boldsymbol{\varphi})_k^s - \frac{\partial(\nabla\boldsymbol{\varphi})}{\partial z} r_e
\end{aligned} \tag{8}$$

By introducing Eq. (8) into the smoothed Galerkin weak form, the smoothed stiffness matrix over the smoothing domain can be modified as

$$\begin{aligned}
\hat{\mathbf{K}}_k &= \frac{A_k}{6}\sum_{i=1}^{6}(\overline{\mathbf{B}}_{ki}^{sc})^{\mathrm{T}}\mathbf{D}(\overline{\mathbf{B}}_{ki}^{sc}) \\
&= \overline{\mathbf{K}}_k + \frac{A_k}{3}(\overline{\mathbf{B}}_k)_x^{\mathrm{T}}\mathbf{D}(\overline{\mathbf{B}}_k)_x + \frac{A_k}{3}(\overline{\mathbf{B}}_k)_y^{\mathrm{T}}\mathbf{D}(\overline{\mathbf{B}}_k)_y + \frac{A_k}{3}(\overline{\mathbf{B}}_k)_z^{\mathrm{T}}\mathbf{D}(\overline{\mathbf{B}}_k)_z
\end{aligned} \tag{9}$$

in which

$$(\overline{\mathbf{B}}_k)_p = \frac{\partial(\overline{\mathbf{B}}_k)}{\partial p} r_e \quad p = x, y, z \tag{10}$$

Obviously, the system stiffness matrix has been strengthened. Besides, it is important to note that the other components of the system equilibrium equations are same to the standard FEM and thus will not be addressed here.

Application to Multiple Physical Problems

Though the NS-FEM possesses many superior properties, it is failed to solve time-dependent physical problems due to the “overly-soft” property. In this section, the SNS-FEM extended to various acoustic problems, heat transfer problems and electromagnetic problems to investigate the performance of the proposed strategy. For comparison, the results of the traditional FEM are also provided.

Acoustic Problems

Computational acoustic governed by Helmholtz equations has been an area of active research for nearly half a century and various methods has been proposed [59]-[60]. The main challenge in solving such problems is the dispersion error which increases dramatically at higher frequency ranges [61]-[62]. Although using refined meshes could alleviate the dispersion effect, a large amount of computational cost will be consumed, which produces great burdens for large scale three-dimensional practical engineering problems. Aimed at this, Wang et al. [52] extended the SNS-FEM to acoustic problems and it turns out that the SNS-FEM can reduce the dispersion error in acoustic problems significantly. Thus, much accurate numerical solutions can be obtained using the SNS-FEM with a rather coarse mesh in the higher frequency range.

Fig. 3 shows a 3D car passenger compartment with different kinds of boundary conditions, namely, Neumann boundary condition and Robin boundary condition. A vibration boundary condition with normal velocity $v_n = 0.01\ \text{m/s}$, i.e., Neumann boundary condition, is imposed on the front panel of the passenger compartment to simulate the vibration from the engine. On the roof of the passenger compartment, sound-absorbing material with admittance coefficient $A_n = 0.00144\ \text{m}^3/(\text{Pa}\cdot\text{s})$ is fixed and regarded as Robin boundary condition. Then, the problem domain is discretized using 4459 nodes and 19278 tetrahedral elements. The average nodal spacing is about 0.11 m which gives an upper frequency limit of 492 Hz based on the well-known "the rule of thumb". Using this mesh, the direct frequency response analysis is conducted. The sound pressure levels (SPL, ref=2×10^{-5} Pa) at the driver's ear point and the passenger's ear point with a full frequency range varies from 1.0 Hz to 500 Hz at intervals of 2.0 Hz are illustrated in Fig. 4. As there is no analytical solution available, the reference solutions are obtained using the FEM with a very fine mesh. It's clear that the SNS-FEM can always provide very accurate results in the full frequency range, while the standard FEM is only available in the lower frequency range. This results prove that the SNS-FEM reduces the dispersion error significantly which indicates that the SNS-FEM can be used to solve higher frequency problems with rather coarse mesh. Thus, the SNS-FEM is very promising in dealing with such mid-frequency acoustic problems.

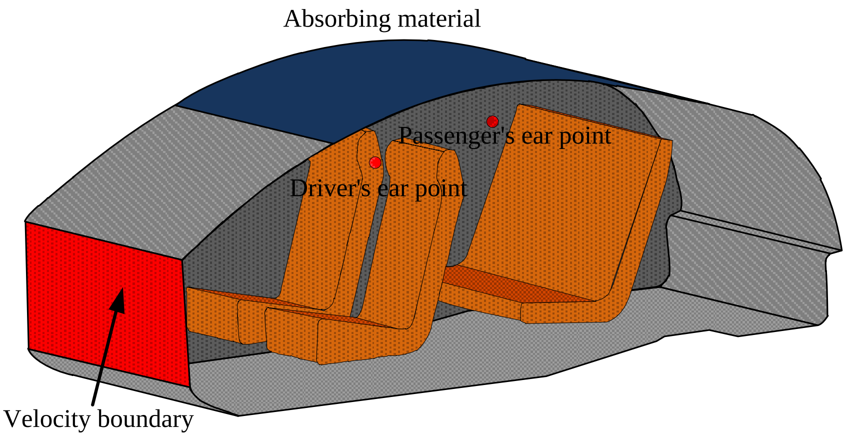
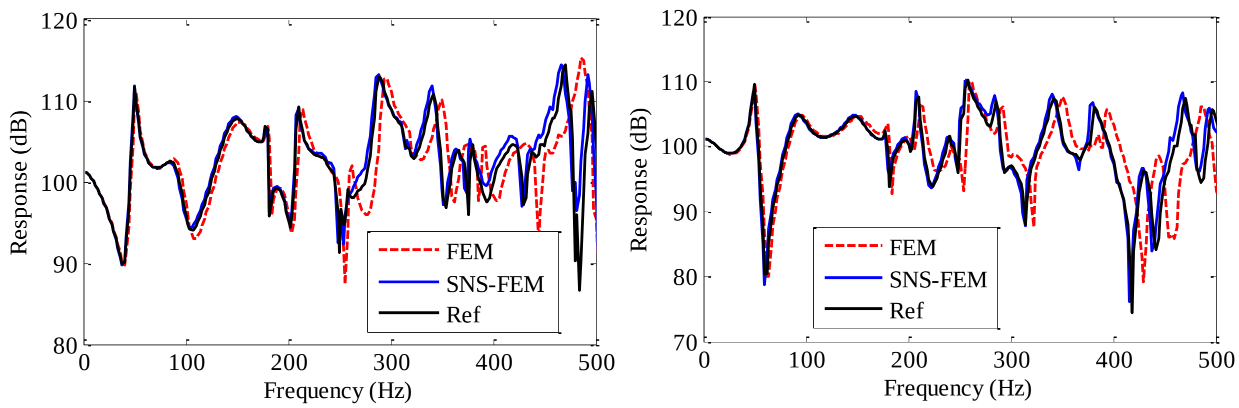


Figure 3. 3D car passenger compartment with different kinds of boundary conditions



(a) Driver's ear point (b) Passenger's ear point

Figure 4. The sound pressure levels at the driver's ear point and the passenger's ear point

Heat Transfer problems

The analysis of heat transfer problems is always of great importance in science and practical engineering and the finite element method has been used for such problems for a long time [63]. However, the poor accuracy and sensitive to mesh distortion properties of the standard FEM have let researchers and engineers down. Cui et al. [56] employed the SNS-FEM to analyze steady and transient heat transfer problems and prove that the SNS-FEM preforms much better in accuracy and efficiency.

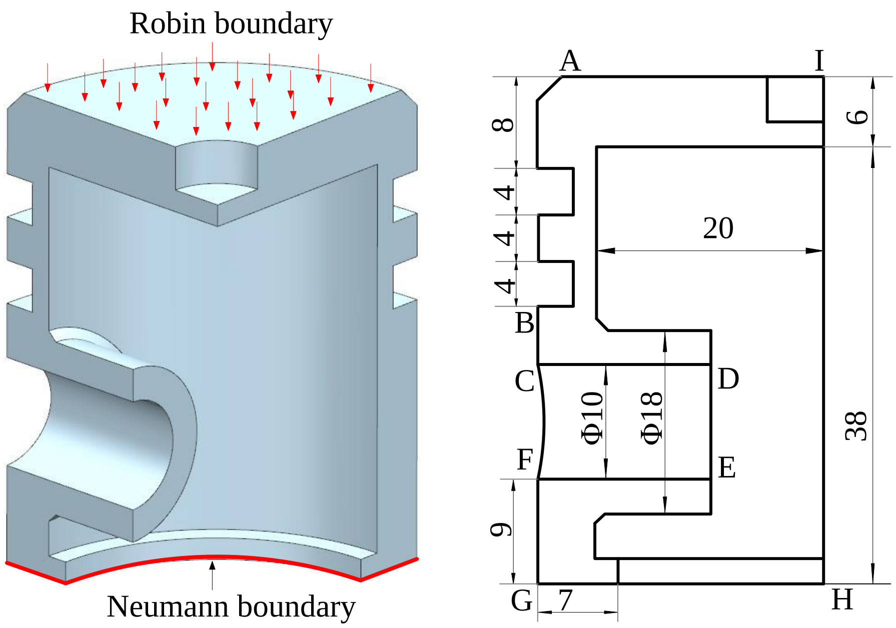


Figure 5. The geometric parameters and the thermal boundary conditions of the piston model

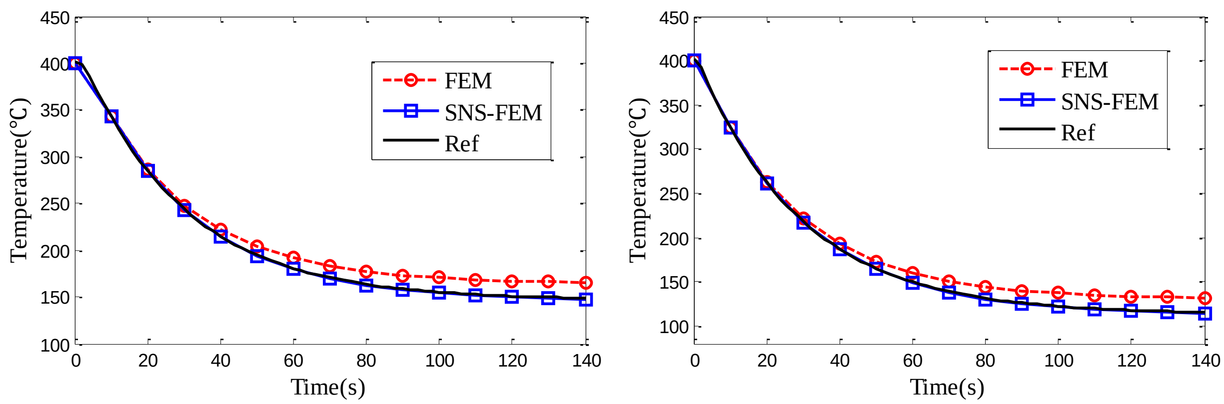


Figure 6. Temperature history of two concerned points

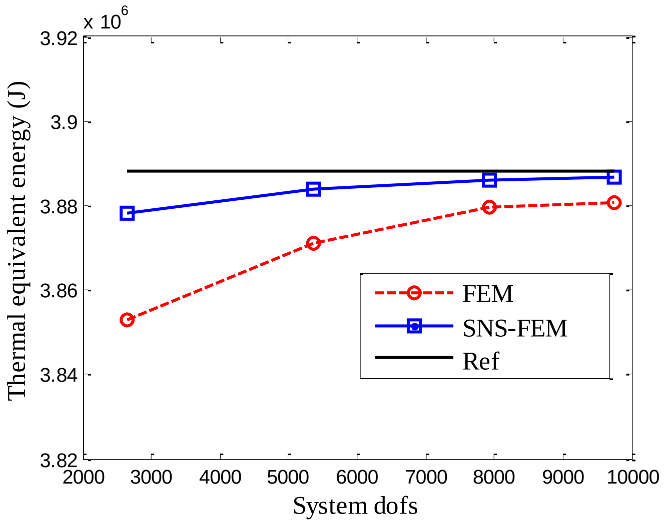


Figure 7. The convergence of the thermal equivalent energy after the system stabilized

A 3D piston model is employed to investigate the performance of the SNS-FEM in analyzing 3D heat transfer problems. The geometric parameters and the thermal boundary conditions of the piston model are shown in Fig. 5. In the computation, the related parameters are taken as the specific heat c=10 J/(kg•℃), the thermal conductivities k_x=k_y= k_z=5 W/(m•℃), the prescribed heat flux q=-2000 W/m2, the convective heat transfer coefficient h= 1000W/(m2•℃), and the temperature of surrounding medium T_a =400 ℃. The reference solutions here are obtained using ABAQUS with a very dense hexahedron mesh. The temperature history of two concerned points D and E are shown in Fig. 6. It's shown that the numerical solutions obtained using the SNS-FEM can always keep in good agreement with the reference. The convergence of the thermal equivalent energy against the number of degrees of freedom after the system stabilized is presented in Fig. 7, from which better results of the SNS-FEM can also be seen obviously.

Electromagnetic Problems

Electromagnetic problems have been studied for decades by researchers and variety of FEMs and meshless methods have been developed for such problems [64]-[65]. The primary field variable for electromagnetic problems is the magnetic vector potential (vector field), which is different from the scalar fields (acoustic pressure, temperature) for acoustic and heat transfer problems. Recently, Feng et al. [57] employed the SNS-FEM to solve static and quasi-static electromagnetic problems and higher accuracy and convergence rate have been achieved.

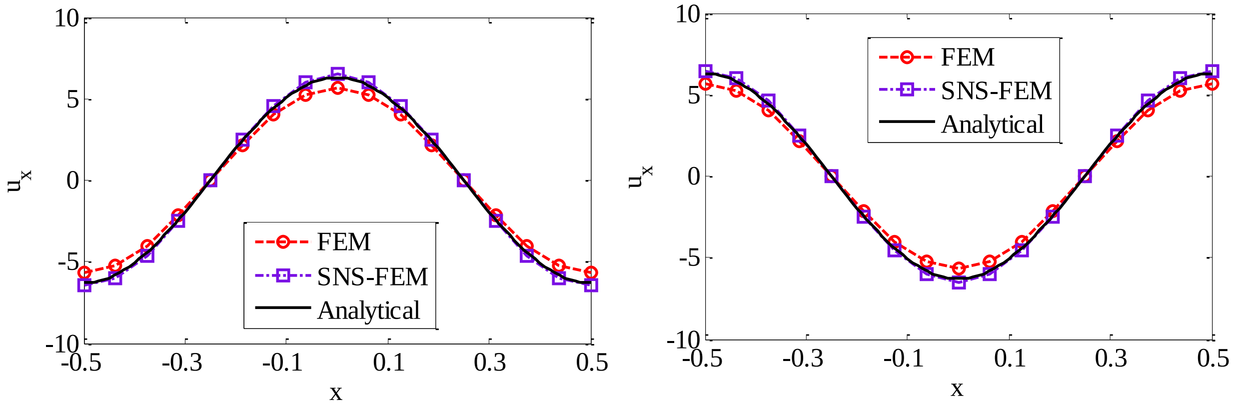
Consider a Poisson problem governed by

$$\nabla^2 u(\mathrm{x},\mathrm{y},\mathrm{z}) = -12\pi^2 \sin(2\pi x)\sin(2\pi y)\sin(2\pi z) \quad (11)$$

with boundary condition

$$u(\mathrm{x},\mathrm{y}) = \sin(2\pi x)\sin(2\pi y)\sin(2\pi z) \quad (12)$$

where $(x, y, z) \in [-0.5, 0.5] \times [-0.5, 0.5] \times [-0.5, 0.5]$. The analytical solution for this problem is available and is shown in Eq. (12). Then, the derivative results of the field variables and the convergence of the solutions have been studied in detail, as illustrated in Fig. 8 and Fig. 9. It can be found that the SNS-FEM can provide much more accurate results and a higher convergence rate compared with the standard FEM.



(a) y=-ly/4, z=-lz/4 (b) y=+ly/4, z=-lz/4

Figure 8. The derivative results of the field variables

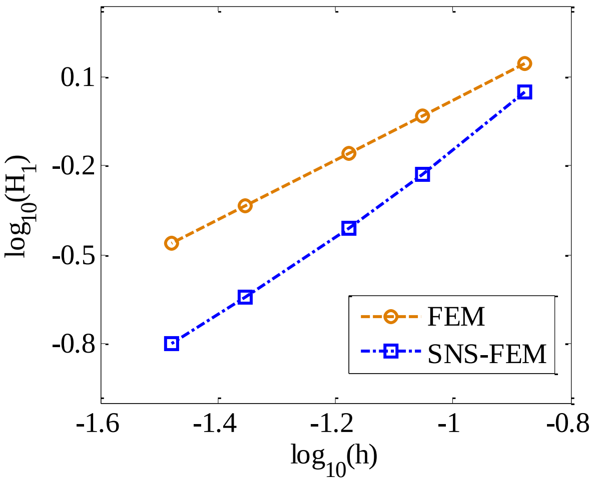


Figure 9. Error estimation in H_1 norm for 3D Poisson equation

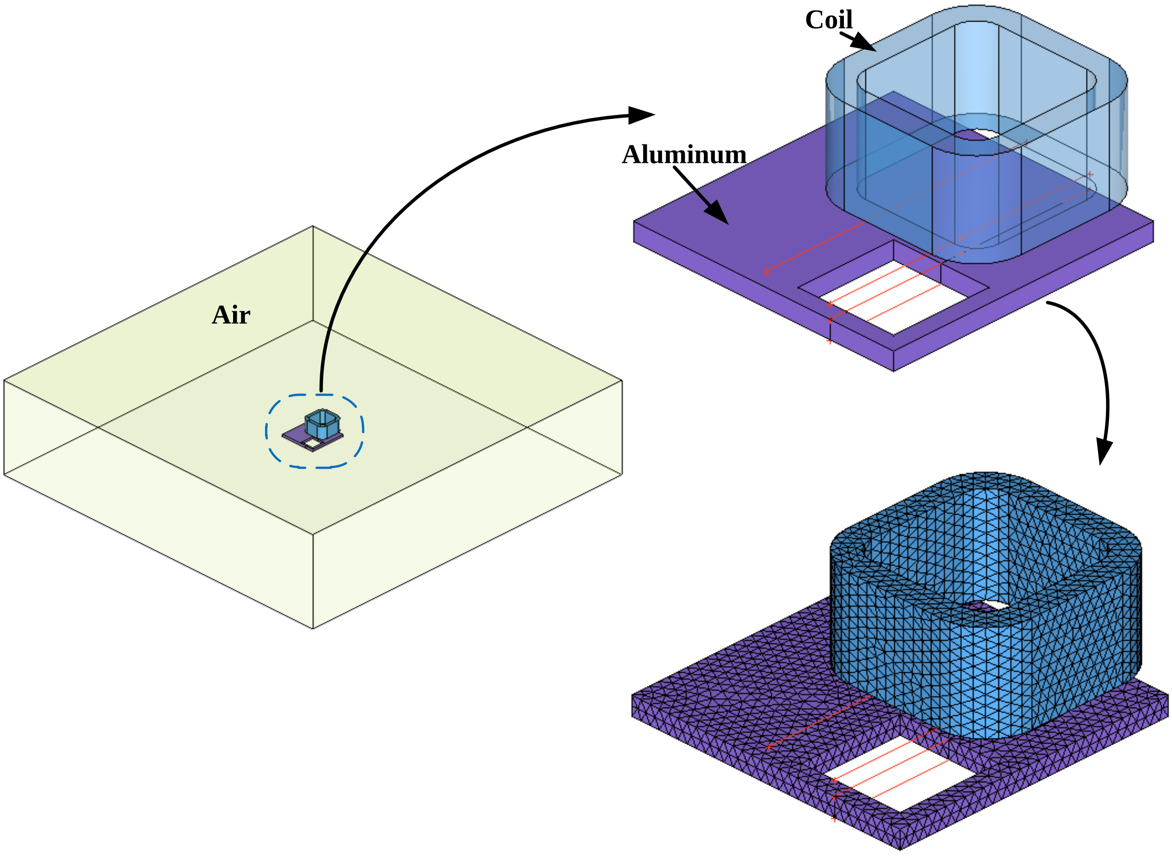
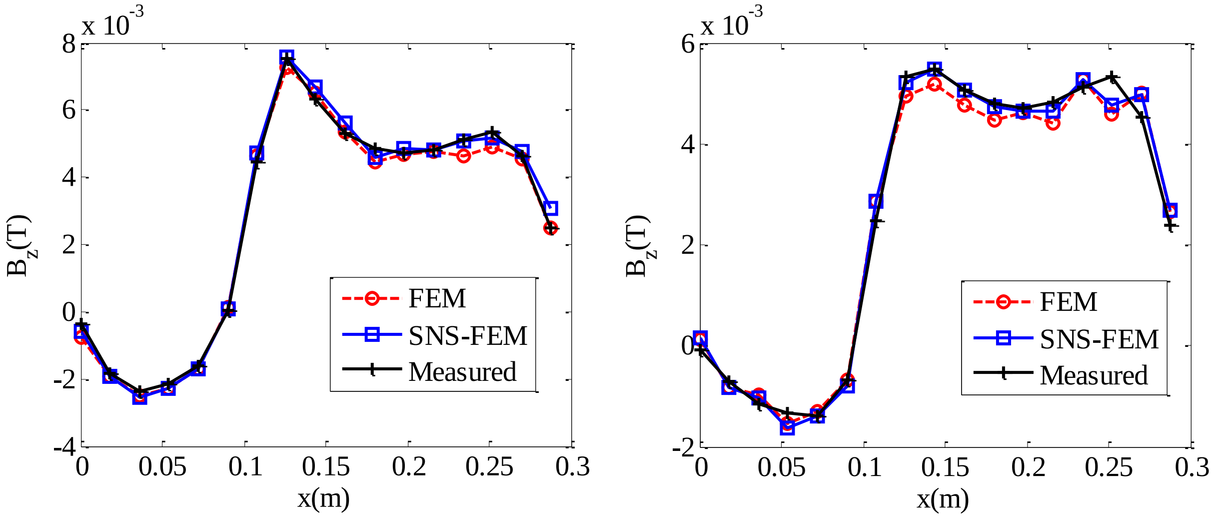


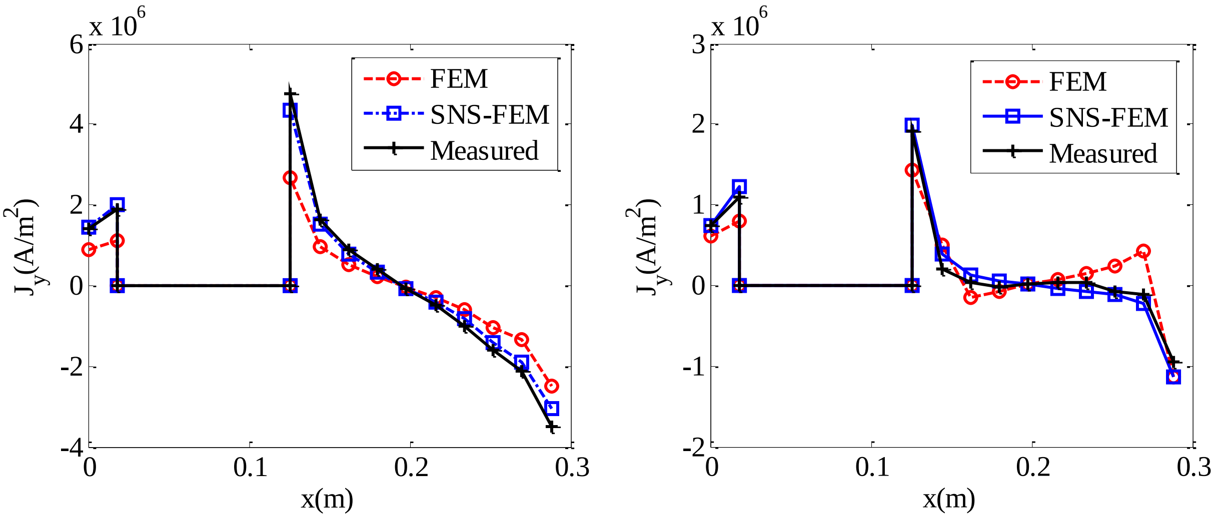
Figure 10. The computational model of TEAM Problem 7

Problem 7 of Testing Electromagnetic Analysis Methods (TEAM) [66]-[67] workshop benchmark problems has also been employed to investigate the application of the SNS-FEM to practical problems. The computational model is shown in Fig. 10, including an asymmetrical conductor with a hole, an exciting coil, and the surrounding air. The conductivity of the conducting plate is 3.537×10^7 S/m and the coil is excited with 2472 Ampere-Turns at 200 Hz. The magnetic flux density distributed along lines at the middle of the exciting coil (A1-B1, y=72 mm) and the conductor (A2-B2, y=144 mm) are shown in Fig. 11. Fig. 12 shows the eddy current density obtained using the SNS-FEM and FEM at the upper surface of the conductor (A3-B3) and the bottom surface of the conductor (A4-B4) together with experimental results. It's obvious that the SNS-FEM can generally achieve quite favorable results and matches well with the experimental ones.



(a) line A1-B1 (b) line A2-B2

Figure 11. Magnetic flux density B_z along certain lines for TEAM Problem 7



(a) line A3-B3 (b) line A4-B4

Figure 12. Eddy current density J_y along certain lines for TEAM Problem 7

Conclusions

A stable node-based smoothed finite element method with a general form for multiple physical problems is presented in this paper. As there is no uncertain parameter introduced in this process, it is a fantastic step forward in the way of stabilizing the NS-FEM. The present scheme is easy to implement and achieves very high accuracy using the simplest linear triangular and tetrahedral elements. The SNS-FEM has been employed to analyze multiple physical problems with good adaptability, such as acoustic, heat transfer and electromagnetic problems. The results demonstrate that the SNS-FEM can generally provide very accurate results and possess super convergence as well as high computational efficiency. All in all, the SNS-FEM is very promising in multiple physical problems and is worthy to be extended to analyze more physical problems.

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Biography

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