Solving nonlinear singular boundary value problems using a newly constructed scaling function

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Abstract

In this paper, a scaling function constructed using special filter coefficients is used for solution of nonlinear singular boundary value problems. The basis functions in interval originated from the newly constructed scaling function are used in function approximation, Galerkin method and iteration approach are used for solution. Some numerical examples are given to demonstrate the validity of the technique. Numerical results prove that the new basis functions have good approximation ability and the present method is very efficient and highly accurate in solving nonlinear singular boundary value problems.

Keywords: filter coefficients, scaling functions, Galerkin method, iteration

Introduction

Many problems in applied mathematics leads to singular boundary value problems which arise in a variety of differential applied mathematics and physics such as gas dynamics, nuclear physics, chemical reaction, studies of atomic structures and atomic calculations. These problems also occur very frequently in the study of electrohydrodynamics and the theory of thermal explosions. There is a vast amount of literature on numerical solutions on singular boundary value problems. Some of the well-known techniques used in solving these problems are finite differences method[1][2], B spline method[3][4], sinc method[5], and reproducing kernel space method[6,7].

Wavelet is a powerful mathematics tool in solving many problems in science and engineering. In recent years, there has been an increasing interest in wavelet-based methods due to their successes in some applications. Wavelet-based numerical method has been developed in recent years. At present, there are mainly three kinds of wavelet-based numerical methods: wavelet finite element method[8][9], wavelet collocation method[10][11]and wavelet-Galerkin method[12][13]. In these methods, wavelet scaling functions and wavelet functions are used as basis functions in functions approximation.

The main aim of this paper is to introduce a new scaling function constructed using special filter coefficients to solve nonlinear singular boundary value problems. The basis functions in interval originated from the new scaling function are directly used to approximate the unknown functions. Using the Galerkin discretization method and iteration approach, the problem will be reduced to a set of algebraic equations. Some numerical examples are given to illustrate the stability and the effectiveness of the present method.

2. The nonlinear singular boundary value problems

In this paper, we consider following nonlinear singular boundary value problems The m degree B-spline is defined as

$$a(x)y'' + \frac{p}{x}y' + b(x)M(y) = f(x) \tag{1}$$

Subject to the boundary conditions

$$y'(0) = 0 \quad y(1) = \beta \tag{2}$$

where $0 < x < 1$, M are nonlinear functions of y , $a(x)$, $b(x)$ and $f(x)$ are given continuous functions, and p, β are finite constants.

3. Functions approximation by new scaling functions

According to the traditional theory of wavelets, the so-called scaling function $\phi(x)$ and wavelet function $w(x)$ both satisfy two-scaling relation

$$\begin{aligned} \phi(x) &= \sum_k h_k \phi(2x-k) \\ w(x) &= \sum_k g_k \phi(2x-k) \end{aligned} \tag{3}$$

The h_k and g_k are called filter coefficients. The dual scaling function $\tilde{\phi}(x)$ and wavelet $\tilde{w}(x)$ also generate a multiresolution analysis. They satisfy refinement relations like (3) with coefficients $\tilde{h}_k$ and $\tilde{g}_k$, respectively. From the theory of wavelets and filter banks, the conditions for perfect reconstruction of dual filters h_k, g_k, $\tilde{h}_k$ and $\tilde{g}_k$ can be stated as

$$\begin{aligned} h(z)\tilde{h}(z^{-1}) + g(z)\tilde{g}(z^{-1}) &= 2 \\ h(z)\tilde{h}(-z^{-1}) + g(z)\tilde{g}(-z^{-1}) &= 0 \end{aligned} \tag{4}$$

where, $h(z)$ denotes the z-transform of h_k

$$h(z) = \sum_k h_k z^{-k} \tag{5}$$

The lifting scheme [14][15]demonstrates that the new filters can be constructed as follows

$$\begin{aligned} g^{\text{new}}(z) &= g(z) + h(z)s(z^2) \\ \tilde{h}^{\text{new}}(z) &= \tilde{h}(z) - g(z)s(z^{-2}) \end{aligned} \tag{6}$$

and the dual lifting scheme can be expressed as

$$\begin{aligned} h^{\text{new}}(z) &= h(z) + g(z)t(z^2) \\ \tilde{g}^{\text{new}}(z) &= \tilde{g}(z) - \tilde{h}(z)t(z^2) \end{aligned} \tag{7}$$

where, $s(z)$ and $t(z)$ are the arbitary Laurent polynomials. The lifting scheme tell us that one can start with the lazy wavelet and use lifting to build filters with particular properties. We can obtain scaling functions and wavelet functions which are suitable for numerical simulation from special filters. In this paper, the filter

$$h(z) = \sum_{k=0}^{5} h_k z^k = 0.05 + 0.3z^{-1} + 0.65z^{-2} + 0.65z^{-3} + 0.3z^{-4} + 0.05z^{-5} \tag{8}$$

is used to construct the scaling functions $\phi(x)$. It is obvious that the support of $\phi(x)$ is

$$\text{supp}\phi(x) = [0,5] \tag{9}$$

From the two-scaling relation (1), the following equation can be obtained

$$\boldsymbol{\phi} = \mathbf{M}\boldsymbol{\phi} \tag{10}$$

where, $\boldsymbol{\phi}$ is a vector

$$\boldsymbol{\phi} = \left[\phi(1), \phi(2), \phi(3), \phi(4)\right]^{\mathrm{T}} \tag{11}$$

$\mathbf{M}$ is a 4 x 4 matrix

$$\mathbf{M} = \begin{bmatrix} 0.3 & 0.05 & 0 & 0 \\ 0.65 & 0.65 & 0.3 & 0.05 \\ 0.05 & 0.3 & 0.65 & 0.65 \\ 0 & 0 & 0.05 & 0.3 \end{bmatrix} \tag{12}$$

From Eq.(10) and additional condition $\phi(1)+\phi(2)+\phi(3)+\phi(4)=1$, the values of $\phi(i), i=1,2,3,4$ can be obtained. Then, $\phi(\frac{k}{2^j}), j,k\in\mathbb{Z}$ can be easily evaluated using the two-scaling relation (3) and $\phi(i)$. Furthermore, the $\phi'(\frac{k}{2^j}),\phi''(\frac{k}{2^j}), j,k\in\mathbb{Z}$ can be obtained by the similar method. Figure 1 shows the scaling function $\phi(x)$ and its first derivative and second derivative. The scaling functions constructed above can be used as basis functions to approximate the function u defined on interval $[0,1]$.

$$u(x)=\sum_{k=-4}^{i-1} c_k\phi_{i,k}(x) \tag{13}$$

where, $\phi_{i,k}(x)=\phi(ix-k)$ and i denotes the scale in approximation. The support of $\phi_{i,k}(x)$ is

$$\mathrm{supp}\phi_{i,k}=[\frac{k}{i},\frac{5+k}{i}] \tag{14}$$

In order to apply boundary conditions effectively, we use the boundary scaling functions in this paper. For $i\geq 5$, the left boundary scaling functions are defined as

$$\phi_{i,-4}^{\mathrm{L}}(x)=\begin{cases}\alpha_{-4}\phi_{i,-4}(x) & 0\leq x<1/i\\ 0 & \text{else}\end{cases} \tag{15}$$

$$\phi_{i,-3}^{\mathrm{L}}(x)=\begin{cases}\phi_{i,-3}(x)-\alpha_{-3}\phi_{i,-4}(x) & 0\leq x<1/i\\ \phi_{i,-3}(x) & 1/i\leq x<2/i\\ 0 & \text{else}\end{cases} \tag{16}$$

$$\phi_{i,-2}^{\mathrm{L}}(x)=\begin{cases}\phi_{i,-2}(x)-\alpha_{-2}\phi_{i,-4}(x) & 0\leq x<1/i\\ \phi_{i,-2}(x) & 1/i\leq x<3/i\\ 0 & \text{else}\end{cases} \tag{17}$$

$$\phi_{i,-1}^{\mathrm{L}}(x)=\begin{cases}\phi_{i,-1}(x)-\alpha_{-1}\phi_{i,-4}(x) & 0\leq x<1/i\\ \phi_{i,-1}(x) & 1/i\leq x<4/i\\ 0 & \text{else}\end{cases} \tag{18}$$

$$\alpha_{-4}=1+\alpha_{-3}+\alpha_{-2}+\alpha_{-1},\ \ \alpha_{-3}=\frac{\phi(3)}{\phi(4)},\ \ \alpha_{-2}=\frac{\phi(2)}{\phi(4)},\ \ \alpha_{-1}=\frac{\phi(1)}{\phi(4)} \tag{19}$$

The right boundary scaling functions $\phi_{i,i+k}^{\mathrm{R}}(x), k=-4,-3,-2,-1$ can be constructed by similar method. Then in approximation Eq.(15), the ordinary scaling functions $\phi_{i,k}(x)$ and $\phi_{i,i+k}(x), k=-4,-3,-2,-1$ are respectively replaced by boundary scaling functions $\phi_{i,k}^{\mathrm{L}}(x)$ and $\phi_{i,i+k}^{\mathrm{R}}(x), k=-4,-3,-2,-1$. For the sake of uniform expressions, the subscript L and R in boundary scaling functions will omit in the following part. The four left boundary scaling functions with $i=5$ are shown in Figure 2.

4. Numerical implementation

In order to solve the Eq.(1), we can approximate the functions $y(x)$ as follows

$$y(x)=\sum_{k=1}^{i+4} c_k\phi_k(x) \tag{20}$$

where $\phi_k(x)$ are corresponding to $\phi_{i,k-5}(x)$ discussed in section 3. Introducing (20) into Eq.(1) and using Galerkin discretization method, we have

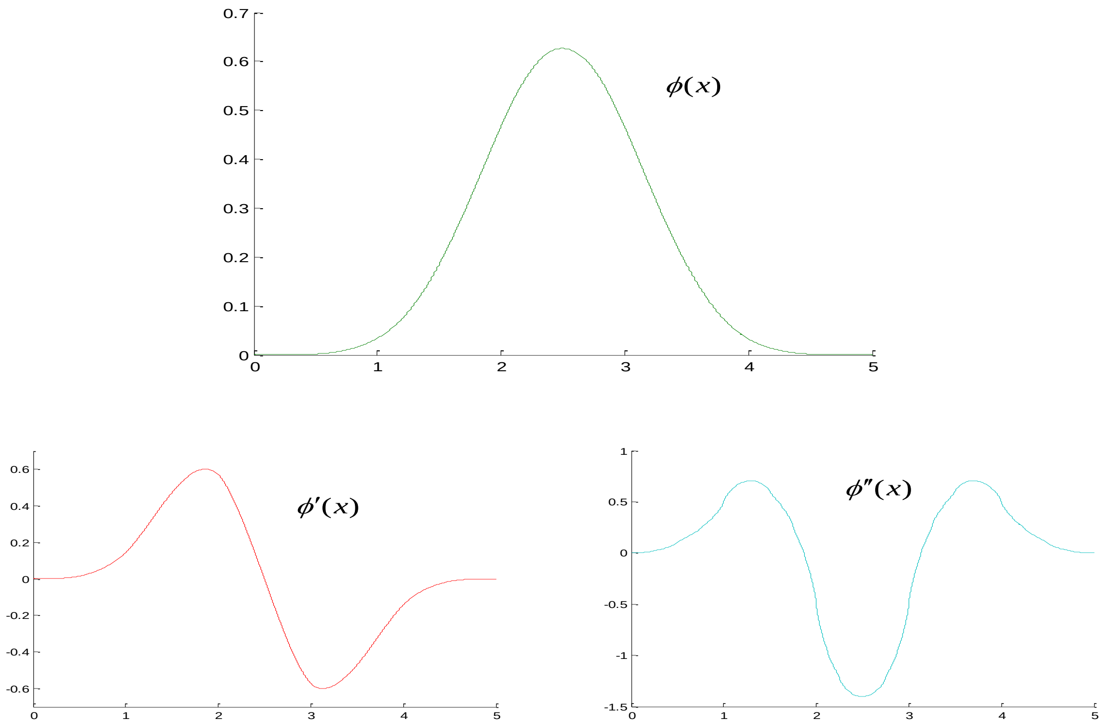

Figure 1. The basic scaling function $\phi(x)$ and its first and second derivative

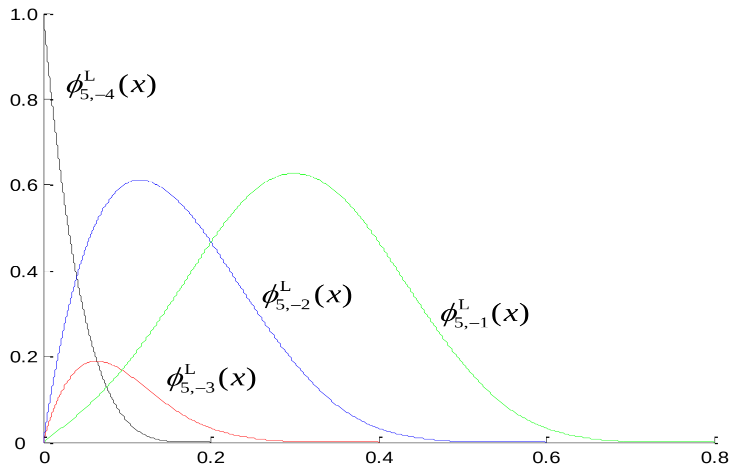

Figure 2. The four left boundary scaling functions with $i=5$

$$\int_0^1 (a(x)\sum_{k=1}^{i+4} c_k \phi_k''(x) + \frac{p}{x}\sum_{k=1}^{i+4} c_k \phi_k'(x))\phi_j(x)dx = \int_0^1 (f(x) - b(x)M(y))\phi_j(x)dx \tag{21}$$

where $j = 1, 2, \cdots i+4$. From (21), we can obtain a set of algebraic equations

$$\mathbf{Kc} = \mathbf{f} \tag{22}$$

where

$$\mathbf{K}(i,j) = \int_0^1 \ (a(x)\phi_i''(x) + \frac{p}{x}\phi_i'(x))\phi_j(x)dx \quad i, j = 1, 2, \cdots i+4 \tag{23}$$

$$\mathbf{c}=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{i+2} \end{bmatrix} \tag{24}$$

$$\mathbf{f}(j)=\int_0^1 (f(x)-b(x)M(y))\phi_j(x)dx\,,\quad j=1,2,\cdots i+4 \tag{25}$$

Because there is nonlinear part in Eq. (22), we should use iteration approach for solution. In this case, we have

$$\mathbf{Kc}^{(n+1)}=\mathbf{f}^{(n)}\quad n=0,1,\cdots \tag{26}$$

where, n is the iteration number and

$$\mathbf{c}^{(0)}=[\mathbf{0}] \tag{27}$$

The computation of $\mathbf{f}^{(n)}$ is as follows

$$\mathbf{f}^{(n)}(j)=\int_0^1 (f(x)-b(x)M(y^{(n)}))\phi_j(x)dx \tag{28}$$

and

$$y^{(n)}(x)=\sum_{k=1}^{i+4} c_k^{(n)}\phi_k(x) \tag{29}$$

5 Numerical examples

In this section, we will apply the present new method to solve some nonlinear singular boundary value problems. The computed results are compared with the exact solutions.

Example1

$$-y''-\frac{2}{x}y'+y^2=x^4-2x^2+7\qquad 0<x<1 \tag{30}$$

Subject to the boundary conditions

$$y'(0)=0\quad y(1)=0 \tag{31}$$

The exact solution of (30) is $y(x)=1-x^2$. Table 1 shows the comparison of exact solution and numerical results of $y(x)$. The scale used in approximation is $i=30$, and the iteration number is $n=8$. It can be found that the results evaluated by present method are highly accurate.

Example2

$$y''+\frac{2}{x}y'+y^5=0\qquad 0<x<1 \tag{32}$$

Subject to the boundary conditions

$$y'(0)=0\quad y(1)=\frac{\sqrt{3}}{2} \tag{33}$$

The exact solution of (32) is $y(x)=1/\sqrt{1+\frac{x^2}{3}}$. Table 2 shows the comparison of exact solution and numerical results of $y(x)$. The scale used in approximation is still $i=30$. The iteration number is $n=10$, and the numerical results are highly accurate.

Table 1. The comparison of exact and numerical results of $y(x)$ for example 1

x	Exact results	Numerical results	Absolute error
0	1.0	0.99993999595962	6.0E-005
0.1	0.99	0.98995337910760	4.7E-005
0.2	0.96	0.95996029190128	4.0E-005
0.3	0.91	0.90996593278944	3.4E-005
0.4	0.84	0.83997127834294	2.9E-005
0.5	0.75	0.74997635685368	2.4E-005
0.6	0.64	0.63998121355224	1.9E-005
0.7	0.51	0.50998591179173	1.4E-005
0.8	0.36	0.35999053476071	9.5E-006
0.9	0.19	0.18999518777879	4.8E-006
1.0	0.0	0.0	0.0

Table 2. The comparison of exact and numerical results of $y(x)$ for example 2

x	Exact results	Numerical results	Absolute error
0	1.0	0.99995346837571	4.7E-005
0.1	0.9983375	0.99829788110237	4.0E-005
0.2	0.9933993	0.99336315548509	3.6E-005
0.3	0.9853293	0.98529691402213	3.2E-005
0.4	0.9743547	0.97432681393413	2.8E-005
0.5	0.9607689	0.96074595606071	2.3E-005
0.6	0.9449112	0.94489331093333	1.8E-005
0.7	0.9271455	0.92713268820308	1.3E-005
0.8	0.9078413	0.90783318696177	8.1E-006
0.9	0.8873565	0.88735271237105	3.8E-006
1.0	0.8660254	0.86602540378444	0.0

Conclusions

In this paper, a scaling function constructed using special filter coefficients is used for solution of nonlinear singular boundary value problems. The basis functions in interval originated from the new scaling function are directly used in function approximation, and the Galerkin discretization method and iteration approach are used for solution. Numerical results demonstrate that the new basis functions are suitable for numerical simulation and the present solution method is very efficient and highly accurate in solving nonlinear singular boundary value problems.

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