# A dual wavelet shrinkage procedure for suppressing numerical oscillation for nonlinear hyperbolic equations

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## Abstract

In the numerical solution for nonlinear hyperbolic equations, numerical oscillation often appears and contaminates the real solution, and sometimes can even make the computation divergent. Using a signal processing approach, a dual wavelet shrinkage procedure is proposed, which allows one to extract the real solution hidden in the numerical solution with oscillation. The dual wavelet shrinkage procedure is introduced after applying the local differential quadrature (LDQ) method, which is a straightforward technique to calculate the spatial derivatives. Results free from numerical oscillation can be obtained, which can not only capture the position of shock and rarefaction waves, but also keep the sharp gradient structure within the shock wave. Three model problems, a one-dimensional dam break flow governed by shallow water equations, and the propagation of a one-dimensional and a two-dimensional shock wave controlled by the Euler equations, are used to confirm the validity of the proposed procedure.

**Key Words:** Wavelet shrinkage; numerical oscillation; shock wave; nonlinear hyperbolic equations; LDQ

## 1. Introduction

Due to the nonlinearity, most problems governed by hyperbolic PDEs in fluid dynamics have to be solved numerically. The main difficulty is that the solution of hyperbolic PDE are bound to develop discontinuities in finite time. A conventional numerical scheme, such as finite difference or finite volume, is directly applied to solve shock wave problem. Two serious problems may appear simultaneously or separately: (1) smearing out shock wave gradually; (2) polluting the shock wave by numerical oscillation. In the recent several decades, people have developed many numerical schemes to maintain the sharp gradient structure of shock wave, meanwhile avoiding the numerical oscillation. Godunov [1959] was credited with introducing the first Riemann solver for the Euler equations, by extending the previous Courant-Isaacson-Reeves (CIR) method to non-linear systems of hyperbolic conservation law. To increase computation's effectiveness, Roe[1981] proposed a second order ROE solver through taking average of square root of density on double sides of a cell. Subsequent works were HLL and HLLC schemes [Toro (1999)]. Efforts in this field led to the proposal of total variance diminishing (TVD) [Loubere *et al* (2014), Toro (1999)] and weighted essentially non-oscillating (WENO) schemes [Liu *et al*(1994), Abgrall (1994)]. Besides these sophisticated schemes, Shyy proposed a non-linear filtering algorithm to eliminate numerical oscillation from second-order central or upwind differencing in calculation of shock wave [Shyy *et al* (1992)]. The filter has proved to be very effective in suppressing oscillation of short wavelength. However, the effect in removing the oscillation of long wavelength is not so promising. Kang introduced a multi-resolution analysis (MRA) for increasing computation's efficiency with preserving the high order numerical accuracy of a conventional solver [Kang *et al* (2014)]. Inspired by their work, we discovered that wavelet's application in suppressing numerical oscillation around the shock wave. Wavelet analysis is characterized by decomposing the signal to be analyzed into multi-scale coefficients, high frequency component is described by coefficients on small scale and low frequency component is described by coefficients on large scale [Gerolymos *et al* (2009), Mallat (1999)]. In this way, shock wave may be maintained and the numerical oscillation around the shock wave may be removed after some special treatment for wavelet coefficients.

In this paper, we propose a dual wavelet shrinkage procedure to suppress numerical oscillation from a straightforward numerical scheme, named localized differential quadrature (LDQ) method, to calculate shock wave problem. LDQ, proposed by Zong [2002], is a high order accurate numerical method The LDQ method was used to solve Riemann problem [Mahdavi *et al* (2012)]. However, high oscillation emerged. A dual wavelet transformation is then applied to process the highly oscillatory results. Three shock wave propagation problems governed by shallow water equations and Euler equations are presented for demonstration. The results are compared to their analytical solutions and very well agreement is achieved.

## 2. A brief introduction of LDQ

Hereinafter, only the main formulas of LDQ are introduced. Details of LDQ and their applications can be referred to Zong [2009]. The first step to LDQ method is to locate the neighborhood of a grid point of interest $x_i$. We use $r_{il}=|x_i-x_l|, i,l=1,\ldots,N$ to denote the distance between any two points in the solution domain. We find the permutation $s(1),s(2),\ldots,s(N)$ to satisfy $r_{is(1)} \le r_{is(2)} \ldots \le r_{is(N)}$.

It is clear that the points falling in the neighborhood of *i*-th point xi are the first m points. Denoting Si=(s(1),s(2),…,s(m)),i=1,…,N, and then Si defines the neighborhood of the grid point of interest. We may get the first and second spatial derivatives at point xi of function f(x) by its neighborhood Si as following:

$$\frac{df(x_i,t)}{dx} \approx \sum_{j\in S_i}^{N} a_{ij} f(x_j,t) \tag{1}$$

$$\frac{d^2 f(x_i,t)}{dx^2} \approx \sum_{j\in S_i}^{N} b_{ij} f(x_j,t) \tag{2}$$

where

$$a_{ij}(x) = \frac{1}{x_j - x_i} \sum_{\substack{k\in S_i \\ k\neq i,j}}^{N} \frac{x_i - x_k}{x_j - x_k}, \; j \in S_i, \, j \neq i \tag{3}$$

$$a_{ii} = -\sum_{\substack{j\in S_i \\ j\neq i}}^{N} a_{ij} \tag{4}$$

$$b_{ij}(x) = 2[a_{ij}a_{ii} - \frac{a_{ij}}{x_i - x_j}], i, j = 1, \ldots, m, i \neq j \quad (5)$$

$$b_{ii}(x) = -\sum_{j \neq i}^{N} b_{ij}(x), i = 1, \ldots, m \quad (6)$$

Multi-dimensional formulas share the same form as in one dimensional because they are independent in each direction.

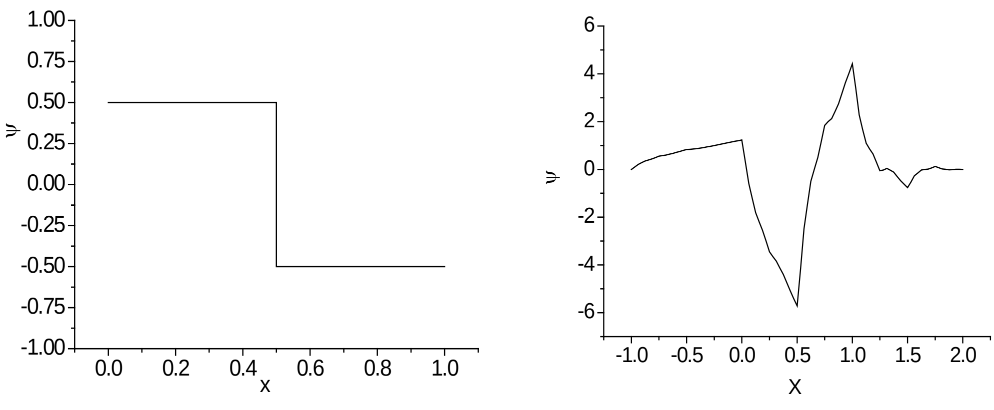
## 3. A dual wavelet shrinkage procedure

Wavelet analysis, introduced by Grossmann & Morlet in the early 1980s, has significantly impacted the signal and image processing [Mallat (1999)]and numerical solving in nonlinear partial differential equations (see Refs. Beylkin [1992], Zong *et al* [2010]，Vasilyev and Paolucci [1996], Vasilyev *et al* [1995] )as well as its application on turbulence(see Refs. Farge *et al* [1999], Farge and Kaiser [2001], Goldstein and Vasilyev [2004], Schneider *et al* [1997], Zong et al [2010] ). Wavelet provides compactly supported, orthogonal or bi-orthogonal basis functions with adjustable smoothness. Due to the compactly supported basis, wavelet coefficients contain local structure information, hence wavelet analysis is a proper mathematical tool to process signal with local structure. Further more, wavelet analysis enables us to obtain multi-scale information of the analyzed signal by introducing scale dilating.

Vanishing moment is an important property of wavelet analysis, which is directly related to wavelet basis functions' smoothness. Wavelet function is called M order vanishing moment if the following relation is satisfied:

$$\int_{-\infty}^{+\infty} t^k \psi(t) dt = 0, k = 0, 1, 2, \cdots, M-1 \quad (7)$$

Daubechies wavelet, one of the most important orthogonal wavelet categories, is widely used in many applications [Daubechies (1988)].The function's smoothness of Daubechies wavelet is adjustable by the vanishing moment order M. Let sup φ=[0,2M-1] be scaling function and sup ψ=[-M+1,M] be wavelet function's supported range. For example, the first or second order of Daubechies wavelet, denoted as DB1 or DB2, are shown in Fig.1.




(a) (b)

**Fig.1 Daubechies wavelet functions of DB1(a) and DB2(b).**

Wavelet bases provide us a multi-scale resolution to express function. We assume function f(x) can be totally approached on the finest scaling index J , i.e. $c_{J,k} = f(x_{J,k})$ and then it can be decomposed in terms of the sum of a series of wavelet bases functions

$$f^J(x) = \sum_k C_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{J-1}\sum_k d_{j,k}\psi_{j,k}(x) \tag{8}$$

The coefficients are obtained by wavelet decomposition formula:

$$c_{j-1,k} = \sum_{l=0}^{L-1} h_l c_{j,l+2k} \tag{9}$$

$$d_{j-1,k} = \sum_{l=0}^{L-1} g_l c_{j,l+2k} \tag{10}$$

For function with most part is well smooth, most wavelet coefficients will be small. Consequently, we can retain a good approximation even discarding a large number of wavelets with small coefficients. More precisely, if we rewrite the approximation as a sum of two terms

$$f^J(x) = f^J_{|d|\ge\varepsilon}(x) + f^J_{|d|\le\varepsilon}(x) \tag{11}$$

Where $\varepsilon$ is a prescribed particular threshold and

$$f^J_{|d|\ge\varepsilon}(x) = \sum_k c_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{J-1}\sum_{|d_{j,k}|\ge\varepsilon} d_{j,k}\psi_{j,k}(x) \tag{12}$$

$$f^J_{|d|<\varepsilon}(x) = \sum_{j=j_0}^{J-1}\sum_{|d_{j,k}|<\varepsilon} d_{j,k}\psi_{j,k}(x) \tag{13}$$

From Vasilyev and Paolucci [1996], the approximation error caused by the significant wavelets, whose coefficient amplitude is above threshold $\varepsilon$ , is bounded by following restriction:

$$\left| f^J(x) - f^J_{|d|\ge\varepsilon}(x) \right| \le C_1\varepsilon \tag{14}$$

and the number of significant wavelet coefficient $K$ is bounded by $\varepsilon$ and wavelet's vanishing moment $N$ as

$$\mathrm{K} \le C_2\varepsilon^{-1/2N} \tag{15}$$

$C_1$, $C_2$ in Eqs.( 14) and (15) depend on wavelet vanishing moment and function *f(x)* .Threshold has two effects: making approximation adaptively and controlling the approximation error globally. The similar situation can be simple extended to multi-dimensional space by tensor product.

For the signal to be analyzed including unknown noise and unknown smoothness structures, how to remove the noise and keep the unknown structure is a complicated problem. Donoho and Johnstone proposed a wavelet shrinkage procedure to extract the structure from noisy sampled data [Donoho and Johnstone (1995)]. We view numerical oscillation as the noise, discontinuity such as shock and rarefaction wave as the signal's structure. Based on this understanding, the wavelet shrinkage procedure is applied to suppress the highly numerical oscillation obtained from the LDQ Method. Indeed, wavelet shrinkage procedure can extract the unknown smoothness structures contaminated by heavy noise. Donoho and Johnstone also addressed that the reconstruction is essentially as smooth as the mother wavelet [Donoho and Johnstone (1995)].This indicates that the reconstructed signal's smoothness is closely related to the adopted wavelet basis functions' smoothness. The reconstructed signal is locally similar as the adopted wavelet basis function, so the optimal wavelet basis should be in the same order smoothness as the real physical signal. Based on the consideration, we propose a dual wavelet shrinkage procedure using DB1 and DB2 for suppressing numerical oscillation obtained from LDQ method. DB1 and DB2 are respectively suitable for shock and rarefaction wave's reconstruction, because DB1 is a sharp jump function and DB2 is a function with one-order smoothness. The dual wavelet shrinkage procedure is proposed as followings.

1) Decomposing the numerical result with highly oscillation obtained from LDQ method via the discrete wavelet transform using DB1 by Mallat algorithm [Goldstein and Vasilyev (2004)], then the wavelet coefficients $d_{j,k}$ is obtained, where $j_0 \leqslant j \leqslant J, 1 \leqslant k \leqslant 2^j$, $j$ is scale index, $j_0$, $J$ represent the largest and smallest scale, respectively.
2) Setting $t_j = \sigma_j \sqrt{2\ln(N_j)/N_j}$ be the threshold value at scale $j$, where $\sigma_j$ is the standard deviation of coefficients $d_{j,k}$, and $N_j$ is the number of wavelet coefficients $d_{j,k}$, i.e. $N_j = 2^j$. A threshold is assigned to each dyadic resolution level for threshold estimates, which is adaptive.
3) Thresholding wavelet coefficients $d_{j,k}$ to get revised coefficients $\hat{d}_{j,k}$ by following

$$\hat{d}_{j,k} = \begin{cases} \operatorname{sgn}(d_{j,k})(|d_{j,k}| - t_j), & if\ |d_{j,k}| \geq t_j \\ 0, & else \end{cases} \tag{16}$$

4) Reconstructing de-noisy data via the wavelet reconstruction transform using the revised wavelet coefficients $\hat{d}_{j,k}$.
5) Repeating the above procedure using DB1. At next time step, the dual wavelet shrinkage procedure is complemented again.

The additional of this computational effort of the overall procedure is only of order $N \cdot log(N)$ as a function of sample size $N$ by Mallat algorithm, which is a fast transform similar as fast Fourier transform, brings little extra computation.

## 4. Numerical tests

### *4.1 One-dimensional dam break flow*

One-dimensional shallow water equations (SWEs) is a typical Riemann problem [Stoker (1986)],

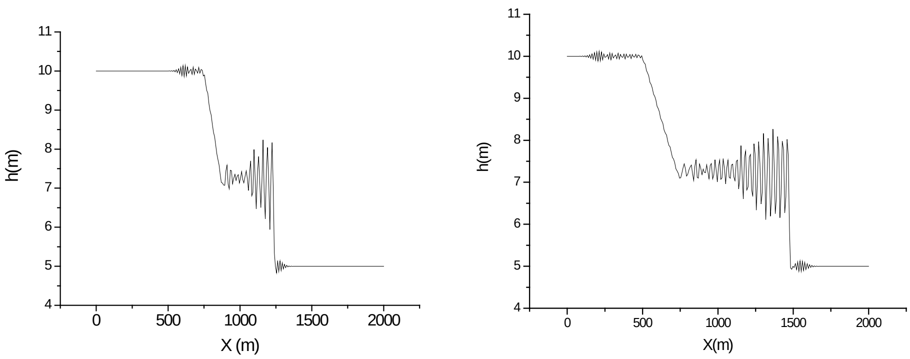
$$U_t + F(U)_x = 0 \tag{17}$$

$$U = \begin{bmatrix} h \\ hu \end{bmatrix}, F(U) = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \end{bmatrix} \quad (18)$$

SWEs describe the flow at time $t \geq 0$ at point $x$, where $h(x,t)$ is the water depth, $u(x,t)$ is the average horizontal velocity and g the gravitational acceleration. A wide variety of physical phenomena are governed by the SWEs, such as tidal flows in coastal water regions, bore wave propagation, flood waves in rivers, surges, dam-break modeling and so on [Delis and Katsaounis (2003)]. Here we use SWEs to model dam-break problem in a channel of length L=2000 m and the initial condition is

$$\begin{aligned} &u(x,0) = 0 \\ &h(x,0) = \begin{cases} 10 & 0 \leq x \leq 1000 \\ 5 & 2000 \geq x > 1000 \end{cases} \end{aligned} \quad (19)$$

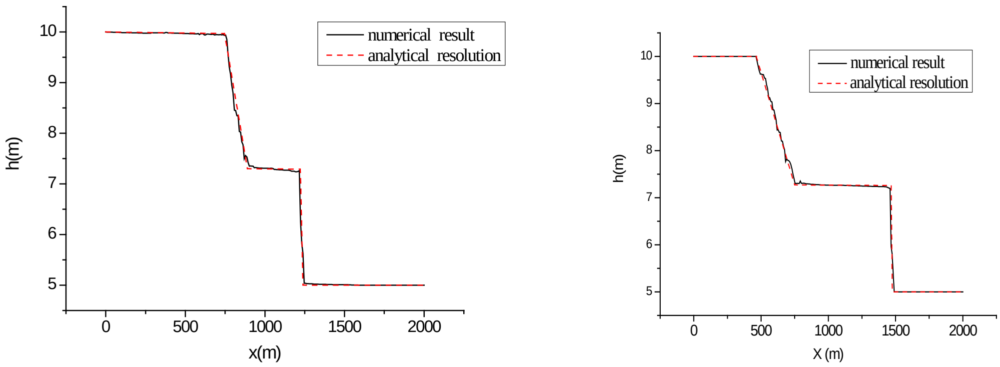
The dam collapses at t =0 and the resulting bow consists of a shock wave traveling downstream and a rarefaction wave traveling upstream. LDQ is applied to the calculation of shock wave tube. However, the viscosity is also included to avoid numerical oscillation in the article [Zhao *et al* (2011)]. In this research, the LDQ method is employed for spatial derivative calculation, where m=5. Uniformly spaced nodes N=256 are set in the channel length. Four-order Runge-Kutta method with time step Δt=0.05s is used for time integration. In order to check the dual wavelet shrinkage procedure's effect, results without wavelet shrinkage procedure at time t=25s and t=50s are shown in Fig.2. Highly numerical oscillatory results are obtained only by LDQ method, especially after the shock wave front. Further more, the oscillation develops along with the time. Real physical solutions are hidden in the highly oscillatory numerical results. The task of the wavelet shrinkage procedure is to reconstruct them by suppressing the numerical oscillation adaptively.




(a) (b)

**Fig.2 Numerical result of water depth only by LDQ at *t=25s (a)* and *t=50s (b)*, respectively**

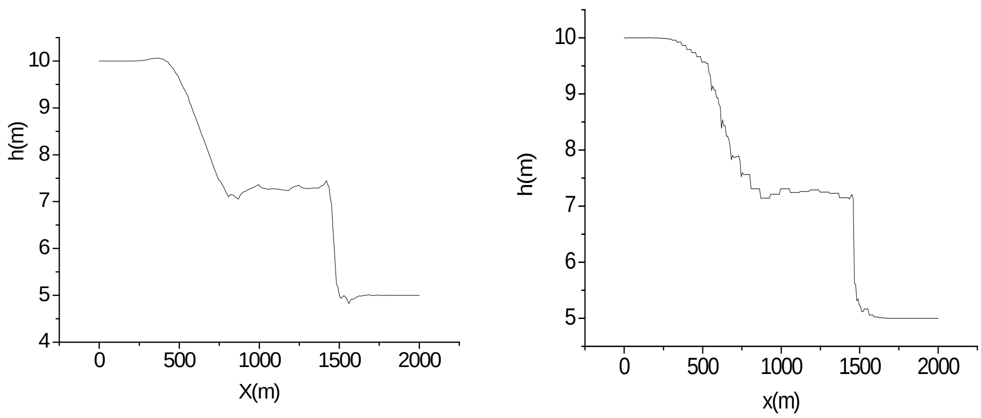
The dual wavelet shrinkage procedure is subsequently implemented after LDQ method to remove the numerical oscillation. Fig.3 shows the numerical result of water depth along the channel at time t=25s and t=50s, which is also compared to the analytical result referred to Stoker [1986]. The highly numerical oscillation is removed out by this dual procedure. Further more, it remains the sharp gradient structure near the shock wave front simultaneously. However, small disturbance is introduced in the domain of rarefaction wave, which could be ignored.




(a) (b)

**Fig.3 Comparison of numerical result for water depth using the dual wavelet shrinkage procedure and analytical result at time *t=25*s (a) and *t=50*s (b), respectively**

In order to examine the dual wavelet shrinkage's superiority over single wavelet shrinkage, we use only DB1 and DB2 in the procedure, respectively. Fig.4 is the result of water depth only using DB2 at time t=50s. As shown in Fig.4(a), DB2 can extract the rarefaction wave's structure well. However, DB2 introduces obvious disturbance with similar shape as DB2 function in the solution. Besides, DB2 makes the shock front excessively smooth, which does not comply with the physical truth. The result obtained from the procedure only using DB1 is shown in Fig.4(b). DB1 can reconstruct shock wave front with sharp gradient structure. But it also introduces many stepped shape error in rarefaction wave and before shock wave front.




(a) (b)

**Fig.4 Water depth along the channel only with DB2 (a) and DB1(b) at *t=50s*, respectively**

The dual procedure combines the respective advantages of DB1 and DB2. DB1 is a discontinuity function, which is the best candidate to capture the shock wave front. DB2, the second order vanishing moment function, is the optimal function to reconstruct the rarefaction wave. The dual wavelet procedure using DB1 and DB2 is a proper combination for processing highly numerical oscillatory results obtained from LDQ method in Riemann problem with shock wave.

### *4.2 One-dimensional shock tube problem*

The Euler equations for one-dimensional unsteady ideal gas flow without heat conduction are given in conservation form.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{20}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 \tag{21}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(u(E + p)) = 0 \tag{22}$$

where $\rho$, $u$, $E$, $p$ is gas' density, velocity, energy per unit volume and pressure, respectively. We need one more equation, i.e the state equation, to close the system.

$$p = (\gamma - 1)(E - \frac{1}{2}\rho u^2) \tag{23}$$

Let ratio of specific heats γ=1.4 and spatial range $-15m \le x \le 15m$. Initial conditions are specified as followings:

$$(\rho, \rho u, E)^T = \begin{cases} (2,0,3), & -15 \le x \le 0, \\ (1,0,1.5), & 15 \ge x > 0 \end{cases} \tag{24}$$

This is a shock tube problem. Setting uniformly spaced nodes with number N=512 in the tube length and m=5 in LDQ method for spatial derivatives' calculation. Fourth-order Runge-Kutta method is used for time integration with time step Δt=0.005s.

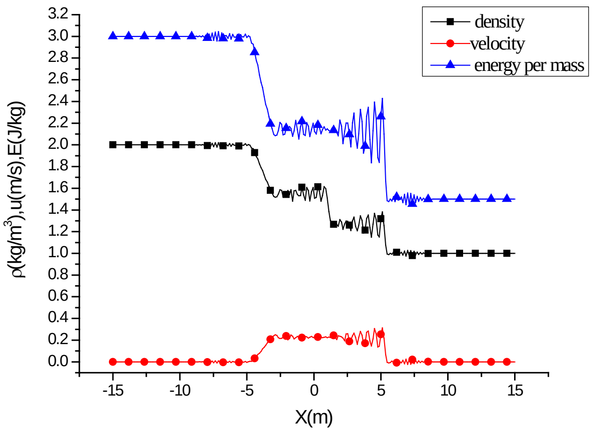



**Fig. 5 Profiles of numerical density, velocity and energy of unit mass only using LDQ at time *t=5s***

In order to reveal the effect of wavelet shrinkage procedure, the numerical results obtained from pure LDQ method is shown in Fig. 5. Intense oscillation appears as in Fig.2.

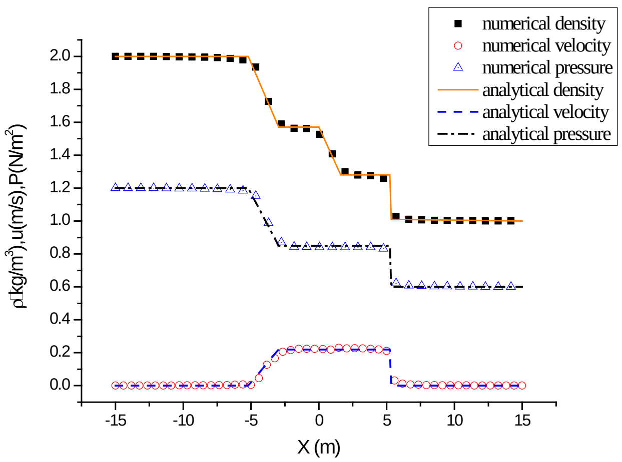



**Fig.6 Comparison of numerical density, velocity and pressure de-noised by the dual wavelet shrinkage procedure and the corresponding analytical solutions at time *t=5s***

After applying the dual wavelet shrinkage procedure to remove the numerical oscillation, the gas's density, velocity and pressure at time t=5s are shown in Fig.6, also including the analytical solutions. As shown in Fig.6, numerical results are very close to their analytical counterparts as referred by Toro [1999].

*4.3 Two-dimensional shock wave propagation*

In order to confirm the procedure's effect in two-dimensional shock wave's computation, we consider two-dimensional Euler equations as following.

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \\ \frac{\partial \rho u}{\partial t} + \frac{\partial \rho uu}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial \rho v}{\partial t} + \frac{\partial \rho vu}{\partial x} + \frac{\partial \rho vv}{\partial y} + \frac{\partial p}{\partial y} = 0 \\ \frac{\partial E}{\partial t} + \frac{\partial (E+p)u}{\partial x} + \frac{\partial (E+p)v}{\partial y} = 0 \\ p = (\gamma - 1)(E - \frac{1}{2}\rho u^2 - \frac{1}{2}\rho v^2) \end{cases} \tag{25}$$

The initial conditions are specified as below:

$$(\rho, \rho u, \rho v, E)^T = \begin{cases} (2,0,0,3), \sqrt{x^2+y^2} \le 10 \\ (1,0,0,1.5), 10 < \sqrt{x^2+y^2} \le 20 \end{cases} \tag{26}$$

Similar calculation parameters are set as in one-dimensional case, i.e. uniformly 512 ×512 nodes in x and y direction respectively; m=5 in LDQ method; and fourth-order Runge-Kutta method for time with time step Δt=0.005s.

A pressure contour at time t=3.75s is shown in Fig.7. Shock wave spreads outward with uniform speed and numerical results keep sharp interface. Pressure keeps the same at the two sides of the contact discontinuity and varies slowly in rarefaction wave region whose position can be seen obviously in Fig.8. It can be clearly seen that the wave spreads uniformly and symmetrically. However, slightly distorted deformation is introduced in radial direction by the effects of the grid which cannot represent a circle by square grid.

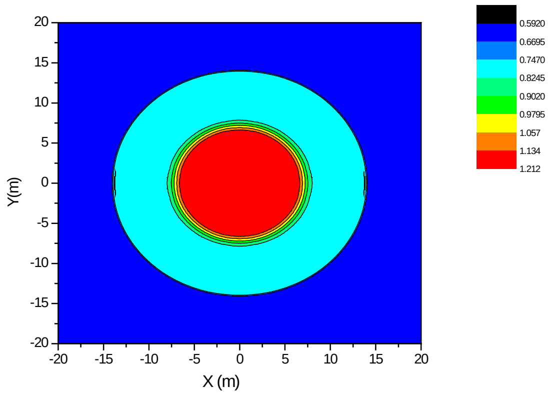



**Fig.7 Pressure contour at *t=3.75*s**

In order to get a clear version of density and pressure, we display their profiles at position of y=0 at time t=3.75s in Fig.8. Symmetry is maintained in the calculation and results with non-oscillation are obtained.

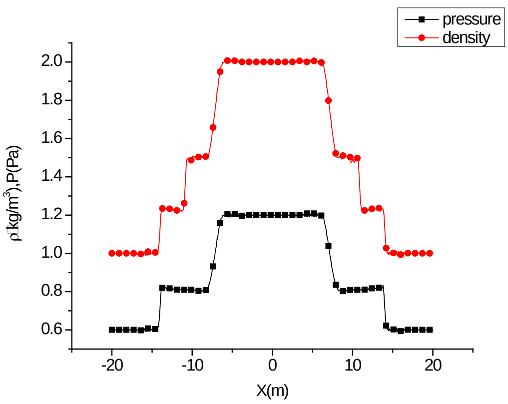



**Fig.8 Profiles of numerical density, velocity components and pressure at position *y=0* at *t=3.75*s**

This numerical model indicates that the dual wavelet shrinkage procedure can be extended to 2D Riemann problem with shock wave and the effect on numerical suppressing is as promising as in 1D case.

## 5. Conclusion

A dual wavelet shrinkage procedure is proposed to suppress the numerical oscillation for nonlinear hyperbolic equations, known as Riemann problem with shock wave in ideal fluid dynamics in this paper. The dual procedure combines the advantage of DB1 and DB2. DB1 is totally discontinuous, which is natural to capture the shock wave front. DB2, the second order vanishing moment function, is the optimal function to reconstruct the rarefaction wave. Based on these coincides, adaptive threshold value is applied to remove the numerical oscillation obtained from LDQ method.

Three numerical tests, *i.e.* one-dimensional water dam breaking problem and one/two-dimensional air shock wave propagating problems, are used to verify the procedure's performance. High quality results are obtained both in capturing discontinuity and suppressing the numerical oscillation. It's demonstrated that the dual wavelet procedure is a proper combination for processing highly numerical oscillation in Riemann problem with shock wave. Compared with the well known Riemann solvers and related complicated numerical schemes, LDQ method is qualified to solve shock wave problem with the aid of the dual wavelet shrinkage procedure.

The method proposed in this article could be applied into ship hydrodynamics numerical methods, such as fluid structure interaction (FSI). Due to most FSI analysis employ the iterative scheme that solvers the solid and fluid problems alternately at each time step, which could accumulate much numerical error and eventually could lead to convergence difficulties.

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