A novel method to improve the multiple-scales solution of the forced strongly nonlinear oscillators

***Hai-En Du, Guo-Kang Er, and Vai Pan Iu**

Department of Civil and Environmental Engineering, University of Macau, Macau SAR, China.

*Presenting author and Corresponding author: haiendu@outlook.com

Abstract

We propose a novel procedure to improve the solution obtained by perturbation methods for analyzing the solutions of strongly nonlinear systems. The multiple-scales method, one of the perturbation method, is widely used in many areas. However, multiple-scales method fails in analyzing the solutions of oscillators if the oscillator nonlinearity is strong. We apply the proposed procedure to improve the multiple-scales method to obtain the optimum solution of the forced oscillator with strongly nonlinear restoring and inertial forces. The solutions obtained from multiple-scales method and the proposed method are examined by the numerical solution obtained from 4^{th}-order Runge-Kutta method. The results show that the proposed method is effective for the oscillators with nonlinear restoring force as well as nonlinear inertial force even if the nonlinearities are strong. Numerical results and comparison show that the proposed method can improve the solution a lot in comparison to the solution obtained by conventional multiple-scales method.

Keywords: Perturbation method; Strong nonlinearity; Nonlinear restoring force; Nonlinear inertial force; Forced Vibration.

Introduction

Strongly nonlinear systems can be found in many structural applications, such as the vibrations of mass-spring system, cable, cantilever with large deflections, etc [1]. Though numerical methods have been widely applied for numerical solutions of nonlinear vibration problems, the solutions of some strongly nonlinear oscillators can still not be completely obtained. Therefore, the studies on the methods for approximate analytical solutions of strongly nonlinear oscillators are attractive. Perturbation methods have been used to obtain the approximate analytical solution for a long time. However, the assumption for perturbation methods limits their applications. Perturbation methods are invalid if the nonlinearity within the system is strong because of the assumption of small perturbation parameter in the system. In order to analyze the vibration of oscillator with strong nonlinearity, some methods have been developed and studied in recent years. They can be categorized as (1) harmonic balance method, (2) variational iteration method, (3) linearized perturbation method, (4) parameter expansion perturbation method, (5) various modified Lindstedt-Poincaré methods and (6) homotopy analysis method. Each of these methods can be applied for obtaining the approximate solutions of a wide class of nonlinear oscillators without introducing a small perturbation parameter as classical perturbation methods do [2]. Wu and Lim proposed a method by combining the linearization of equation of motion and harmonic balance method to analyze the free vibration of an ordinary differential equation with odd nonlinear restoring force [3]. Cheung and Iu applied the harmonic balance method to analyze the forced vibration of a dynamical system with quadratic and cubic nonlinearities [4]. Hu applied a modified iteration procedure to a quadratic nonlinear oscillator (QNO) and obtained an improved solution in comparison to those obtained by the first-order harmonic balance method [5].

Shakeri and Dehghan adopted the variational iteration method to solve the Klein-Gordon equation and it shows that the solution converges fast [6]. Marinca and Herisanu proposed a perturbation technique by combining the iteration methods and the solution obtained by this new method agrees well with exact solution [7]. The linearized perturbation technique is applied to a Duffing oscillator with 5th-order nonlinearity [8]. Xu applied He's parameter-expanding method (PEM) to determine the limit cycles of the strongly nonlinear oscillators [9]. With this method, a strongly nonlinear oscillator with large perturbation parameter is transformed into an oscillator with small parameter. Chen et al. proposed a modified Lindstedt-Poincaré method for the analytical approximate solution of limit cycles in three-dimensional nonlinear autonomous dynamical systems [10]. In 2009, Pakdemirli proposed a method named multiple-scales Lindstedt-Poincaré (MSLP) method by combining the multiple-scales (MS) method and Lindstedt-Poincaré (LP) method. This method has been applied to analyze the free vibration of three oscillators which are the damped linear oscillator, undamped Duffing oscillator and damped Duffing oscillator [11]. Later, the MSLP method was extended to analyze the forced vibration of strongly nonlinear Duffing oscillator [12]. Liao proposed an optimal homotopy analysis method by introducing a two-parameter family equation to find the fastest convergence solution to the Blasius boundary-layer flows problem in fluid mechanics [13]. Razzak and Molla combined the general Struble's technique and homotopy perturbation method to analyze damped and driven strongly nonlinearDuffing oscillator and strongly nonlinear van der Pol oscillator with damping [14].

Since the validity condition for perturbation method to give a valid solution is that the ratio of the amplitude of $O(\varepsilon^1)$ solution and that of $O(\varepsilon^0)$ solution is much less than unity [15], the method proposed in this paper is based on the objective that the ratio of the amplitude of $O(\varepsilon^1)$ solution and that of $O(\varepsilon^0)$ solution is minimized. To do so, an equivalent oscillator is formulated by splitting the parameters in nonlinear restoring force and nonlinear inertial force. The introduced unknown nonlinearity parameters can be determined with the objective that the ratio of the amplitude of $O(\varepsilon^1)$ solution and that of $O(\varepsilon^0)$ solution is minimized. An oscillator is analyzed by multiple-scales (MS) method and the proposed method which is named parameter-split-multiple-scales (PSMS) method. The solutions obtained by these methods are compared to the numerical solutions obtained by the 4th-order Runge-Kutta method. The accuracy and the effectiveness of PSMS method are examined by numerical analysis.

Procedures for optimizing the solution obtained by the multiple-scales method

Considering the following nonlinear oscillator

$$\ddot{y}+c\varepsilon^2\dot{y}+\omega_0^2 y+\varepsilon g\left(y,\dot{y},\ddot{y}\right)=F\varepsilon^2\cos\left(\Omega t\right) \tag{1}$$

where y is displacement, t is time, c is damping coefficient, ω_0 is the natural frequency of the oscillator, ε is perturbation parameter, F is excitation amplitude, Ω is excitation frequency and $g\left(y,\dot{y},\ddot{y}\right)$ is a nonlinear function and given as

$$g\left(y,\dot{y},\ddot{y}\right)=\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta^{(i,j,k)}y^i\dot{y}^j\ddot{y}^k \tag{2}$$

where $\eta^{(i,j,k)}$ are nonlinear parameters which reflect the degrees of nonlinearity and $\sum i+j+k\geq 2$. The nonlinear parameters $\eta^{(i,j,k)}$ are split and expressed by two terms as follows.

$$\eta^{(i,j,k)} = \eta_1^{(i,j,k)} + \eta_2^{(i,j,k)}\varepsilon \tag{3}$$

Then, Eq. (1) is written as

$$\ddot{y} + c\varepsilon^2\dot{y} + \omega_0^2 y + \varepsilon g_1\left(y,\dot{y},\ddot{y}\right) + \varepsilon^2 g_2\left(y,\dot{y},\ddot{y}\right) = F\varepsilon^2\cos\left(\Omega t\right) \tag{4}$$

where $$g_1\left(y,\dot{y},\ddot{y}\right) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_1^{(i,j,k)} y^i \dot{y}^j \ddot{y}^k \qquad g_2\left(y,\dot{y},\ddot{y}\right) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_2^{(i,j,k)} y^i \dot{y}^j \ddot{y}^k \tag{5}$$

In the analysis with perturbation method, the response of the oscillator is assumed to be

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + O\left(\varepsilon^3\right) \tag{6}$$

Substituting Eq. (7) into Eq. (4) leads to

$$\begin{aligned}
&\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2 + c\varepsilon^2\dot{y}_0 + \omega_0^2\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2\right) \\
&+\varepsilon\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_1^{(i,j,k)}\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2\right)^i\left(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2\right)^j\left(\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2\right)^k \\
&+\varepsilon^2\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l}\eta_2^{(i,j,k)}\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2\right)^i\left(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2\right)^j\left(\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2\right)^k \\
&= F\varepsilon^2\cos\left(\Omega t\right)
\end{aligned} \tag{7}$$

Equating the coefficients of ε^s (s=0, 1, 2) to zero and eliminating the secular terms one can obtain an approximate steady-state response to the oscillator in the form of

$$y = A\cos\left(\Omega t - \gamma\right) + Y_1\cos\left[3\left(\Omega t - \gamma\right)\right] + Y_2\cos\left[5\left(\Omega t - \gamma\right)\right] \tag{8}$$

in which A is the steady-state response amplitude, γ is the steady-state phase angle, Y_1 and Y_2 are the amplitudes of $O(\varepsilon^1)$ solution and $O(\varepsilon^2)$ solution, respectively. They are also the functions of $\eta_1^{(1,1,0)},\ldots,\eta_2^{(n,m,l)}$. Then the values of $\eta_1^{(i,j,k)}$ and $\eta_2^{(i,j,k)}$ are determined by a numerical iteration procedure.

Damped and driven Duffing equation with nonlinear inertial force

Consider a damped and driven Duffing equation with nonlinear inertial force as follows.

$$\ddot{q} + 2u\varepsilon^2\dot{q} + \omega_0^2 q + \alpha\varepsilon q\dot{q}^2 + \alpha\varepsilon q^2\ddot{q} + \beta\varepsilon q^3 = F\varepsilon^2\cos\left(\Omega t\right) \tag{9}$$

which can find its applications in the nonlinear vibrations of cantilever with large deflection. The nonlinear parameter α and β are split by

$$\alpha = \alpha_1 + \alpha_2\varepsilon \tag{10}$$

$$\beta = \beta_1 + \beta_2\varepsilon \tag{11}$$

The oscillator response is expressed as

$$q = q_0\left(T_0,T_1,T_2\right) + \varepsilon q_1\left(T_0,T_1,T_2\right) + \varepsilon^2 q_2\left(T_0,T_1,T_2\right) + O\left(\varepsilon^3\right) \tag{12}$$

where T_0, T_1 and T_2 are different time scales with multiple-scales method which are given by

$$T_0 = t, T_1 = \varepsilon t, T_2 = \varepsilon t^2. \tag{13}$$

By chain rule, the operators of time derivatives are

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \tag{14}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 \left(D_1^2 + 2D_0 D_2\right) + \ldots, \tag{15}$$

where $D_n = \partial / \partial T_n$ and $D_n^2 = \partial^2 / \partial T_n^2$. Substituting Eq. (10)-(15) into Eq. (9) and setting the coefficients of $\varepsilon^m \left(m = 0, 1, 2\right)$ to zero lead to the following equations.

$$O\left(\varepsilon^0\right): \quad D_0^2\left(q_0\right) + \omega_0^2 q_0 = 0, \tag{16}$$

$$O\left(\varepsilon^1\right): \quad D_0^2\left(q_1\right) + \omega_0^2 q_1 = -2D_0 D_1\left(q_0\right) - \alpha_1 q_0 \left[D_0\left(q_0\right)\right]^2 - \alpha_1 q_0^2 D_0^2\left(q_0\right) - \beta_1 q_0^3 \tag{17}$$

$$O\left(\varepsilon^2\right): \quad \begin{aligned} &D_0^2\left(q_2\right) + \omega_0^2 q_2 = -2D_0 D_1\left(q_1\right) - D_1^2\left(q_0\right) - \beta_2 q_0^3 - 2D_0 D_2\left(q_0\right) \\ &-2uD_0\left(q_0\right) - 3\beta_1 q_0^2 q_1 - \alpha_1 q_1 \left[D_0\left(q_0\right)\right]^2 - \alpha_2 q_0 \left[D_0\left(q_0\right)\right]^2 - \alpha_2 q_0^2 D_0^2\left(q_0\right) \\ &-2\alpha_1 q_0 D_0\left(q_0\right) D_0\left(q_1\right) - 2\alpha_1 q_0 D_0\left(q_0\right) D_1\left(q_0\right) - \alpha_1 q_0^2 D_0^2\left(q_1\right) \\ &-2\alpha_1 q_0^2 D_0 D_1\left(q_0\right) - 2\alpha_1 q_0 q_1 D_0^2\left(q_0\right) + F\cos\left(\Omega t\right). \end{aligned} \tag{18}$$

The solution of the $O\left(\varepsilon^0\right)$ equation is

$$q_0 = C\left(T_1, T_2\right) e^{i\omega_0 T_0} + \overline{C}\left(T_1, T_2\right) e^{-i\omega_0 T_0} \tag{19}$$

where C is a function of time scales T_1 and T_2 which can be determined by omitting the secular terms in the $O\left(\varepsilon^1\right)$ equation in the following. Substituting Eq. (19) into the righthand side of the $O\left(\varepsilon^1\right)$ equation and eliminating the secular terms yield

$$2i\omega_0 D_1\left(C\right) + 3\beta_1 C^2 \overline{C} - 2\alpha_1 \omega_0^2 C^2 \overline{C} = 0 \tag{20}$$

and

$$q_1 = \Lambda e^{3i\omega_0 T_0} + \overline{\Lambda} e^{-3i\omega_0 T_0}, \tag{21}$$

in which

$$\Lambda = \frac{\beta_1 C^3}{8\omega_0^2} - \frac{\alpha_1 C^3}{4}. \tag{22}$$

Substituting the expressions of q_0 and q_1 into the $O\left(\varepsilon^2\right)$ equation, eliminating the secular terms, and using the expression $\Omega = \omega_0 + \varepsilon^2 \sigma$ where σ is a detuning parameter that can be determined if Ω is given, it gives

$$q_2 = \Gamma_1 e^{3i\omega_0 T_0} + \Gamma_2 e^{5i\omega_0 T_0} + \overline{\Gamma}_1 e^{-3i\omega_0 T_0} + \overline{\Gamma}_2 e^{-5i\omega_0 T_0}, \tag{23}$$

in which

$$\Gamma_1 = \frac{9\alpha_1 C^4 \overline{C}}{16} - \frac{\alpha_2 C^3}{4} + \frac{\beta_2 C^3}{8\omega_0^2} - \frac{\alpha_1 \beta_1 C^4 \overline{C}}{8\omega_0^2} - \frac{21\beta_1^2 C^4 \overline{C}}{64\omega_0^4}, \tag{24}$$

$$\Gamma_2 = \frac{3\alpha_1^2 C^5}{16} - \frac{\alpha_1 \beta_1 C^5}{8\omega_0^2} + \frac{\beta_1^2 C^5}{64\omega_0^4}. \tag{25}$$

$D_2\left(C\right)$ is selected to eliminate the secular terms and expressed as

$$D_2\left(C\right) = \frac{Fe^{i\sigma T_2}}{4i\omega_0} - uC - \frac{9\alpha_1^2 \omega_0 C^3 \overline{C}^2}{4i} + \frac{9\alpha_1 \beta_1 C^3 \overline{C}^2}{4i\omega_0} + \frac{15\beta_2^2 C^3 \overline{C}^2}{16i\omega_0^3} \tag{26}$$

The time derivative of C can be expressed as

$$\frac{dC}{dt} = \varepsilon D_1\left(C\right) + \varepsilon^2 D_2\left(C\right) + O\left(\varepsilon^3\right). \tag{27}$$

The polar form of C is assumed to be

$$C = \frac{1}{2} A e^{ib}, \tag{28}$$

where A is the response amplitude and b is the phase of oscillator response. Substituting Eqs. (20), (26) and (28) into Eq. (27) and separating the real and imaginary parts yield

$$\dot{A} = \frac{F\varepsilon^2}{2\omega_0} \sin\gamma - uA\varepsilon^2 \tag{29}$$

and

$$\dot{\gamma} = \varepsilon^2 \sigma + \varepsilon \left(\frac{A^2 \alpha_1 \omega_0}{4} - \frac{3A^2 \beta_1}{8\omega_0} \right) + \varepsilon^2 \begin{pmatrix} \dfrac{F\cos\gamma}{2A\omega_0} - \dfrac{9A^4 \alpha_1^2 \omega_0}{64} + \dfrac{9A^4 \alpha_1 \beta_1}{64\omega_0} \\ + \dfrac{15A^4 \beta_1^2}{256\omega_0^3} + \dfrac{\alpha_2 A^2 \omega_0}{4} - \dfrac{3\beta_2 A^2}{8\omega_0} \end{pmatrix}, \tag{30}$$

where $\gamma = \sigma T_2 - b$.

For steady state, $\dot{A}$ and $\dot{\gamma}$ are equal to zero. Then the frequency response curve can be obtained by eliminating γ and σ in Eq. (30). The relation between the excitation frequency and the response amplitude at steady state can be obtained to be

$$\begin{aligned} \Omega = \omega_0 &- \frac{A^2 \alpha_1 \omega_0 \varepsilon}{4} + \frac{3A^2 \beta_1 \varepsilon}{8\omega_0} + \frac{9A^4 \alpha_1^2 \omega_0 \varepsilon^2}{64} - \frac{9A^4 \alpha_1 \beta_1 \varepsilon^2}{64\omega_0} - \frac{15A^4 \beta_1^2 \varepsilon^2}{256\omega_0^3} \\ &- \frac{\alpha_2 A^2 \omega_0 \varepsilon^2}{4} + \frac{3\beta_2 A^2 \varepsilon^2}{8\omega_0} \mp \frac{F\varepsilon^2}{2A\omega_0} \sqrt{1 - \frac{4A^2 u^2 \omega_0^2}{F^2}}. \end{aligned} \tag{31}$$

The approximate response of the oscillator can be expressed as

$$q = A\left\{\cos\left(\Omega t - \gamma\right) + X_1 \cos\left[3\left(\Omega t - \gamma\right)\right] + X_2 \cos\left[5\left(\Omega t - \gamma\right)\right]\right\}, \tag{32}$$

in which

$$X_1 = \frac{9A^4\alpha_1^2\varepsilon^2}{256} - \frac{A^2\alpha_1\varepsilon}{16} - \frac{21A^4\beta_1^2\varepsilon^2}{1024\omega_0^4} + \frac{A^2\beta_1\varepsilon}{32\omega_0^2} - \frac{\alpha_2A^2\varepsilon^2}{16} + \frac{\beta_2A^2\varepsilon^2}{32\omega_0^2} - \frac{A^4\alpha_1\beta_1\varepsilon^2}{128\omega_0^2} \tag{33}$$

and

$$X_2 = \frac{3A^4\alpha_1^2\varepsilon^2}{256} + \frac{A^4\beta_1^2\varepsilon^2}{1024\omega_0^4} - \frac{A^4\alpha_1\beta_1\varepsilon^2}{128\omega_0^2}. \tag{34}$$

Due to the relations given by Eqs. (10) and (11), two of the parameters α_1, α_2, β_1 and β_2 are independent if the values of α and β are given. Consider the parameters α_1 and β_1 as independent parameters. In order get the optimum solution, the values of α_1 and β_1 are determined such that the absolute value of X_1 is minimized.

Case 1: $\alpha = 0$

When $\alpha = 0$, the considered oscillator can be regarded as a damped and driven Duffing oscillator which can be found in many applications such as the forced vibrations of pendulum, isolator, electrical circuit [1].

The frequency response curves obtained by the proposed method and the multiple-scales method are compared to the frequency response curve obtained by the fourth-order Runge-Kutta method to examine for the effectiveness of the methods.

The parameters of nonlinear oscillators are listed in Table 1.

Table 1. Oscillator parameters

Oscillator	ε	ω_0	u	α	β	F
1	0.1	1	2	0	10	30
2	0.1	1	2	0	0.1	30
3	0.1	1	2	0	10	30

The frequency response curves of oscillators 1, 2 and 3, obtained by the proposed method, the multiple-scales method and the numerical simulation are presented in Figs. 1-3, respectively.

Case 2: $\alpha \neq 0$

When $\alpha \neq 0$, the considered oscillator can be regarded as a damped and driven Duffing oscillator with nonlinear inertial forces ($q^2\ddot{q}$ and $q\dot{q}^2$) which can be found in the forced vibrations of beams [16].

The frequency response curves obtained by the proposed method and the multiple-scales method are compared to the frequency response curve obtained by the fourth-order Runge-Kutta method to examine the effectiveness of the methods.

The parameters of the nonlinear oscillators are listed in Table 2.

Table 2. Oscillator parameters

Oscillator	ε	ω_0	u	α	β	F
4	0.1	1	2	0.1	10	30
5	0.1	1	2	2	0.1	30
6	0.1	1	2	2	10	30

The frequency response curves of oscillators 4, 5 and 6, obtained by the proposed method, the multiple-scales method and the numerical simulation are presented in Figs. 4-6, respectively.

Conclusions

A novel method named parameter-split-multiple-scales method is proposed to improve the solution obtained by perturbation method based on the objective that the ratio of the amplitude of $O(\varepsilon^1)$ solution and that of $O(\varepsilon^0)$ solution is minimized. The forced vibration of an oscillator with strongly nonlinear restoring and inertial forces is analyzed by the proposed PSMS method, MS method and 4^{th}-order Runge-Kutta method. We have first studied the case that $\alpha = 0$ to examine the validity of the proposed method when nonlinear restoring force is large. After that, we have studied the oscillator with nonlinear restoring and inertial forces ($\alpha \neq 0$). The results show that the proposed method works for the oscillators with strongly nonlinear restoring force and/or strongly nonlinear inertial force. It can improve make the solutions improved a lot compared to the conventional multiple scales method.

Acknowledgement

The results presented in this paper were obtained under the supports of the Research Committee of University of Macau (Grant No. MYRG2014-00084-FST) and the Science and Technology Development Fund of Macau (Grant No. 043/2013/A).

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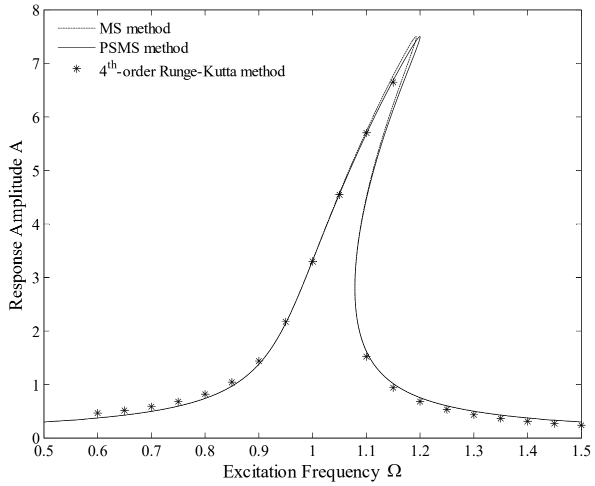

Figure 1. FRCs of oscillator 1 by the proposed method, the MS method and numerical simulation.

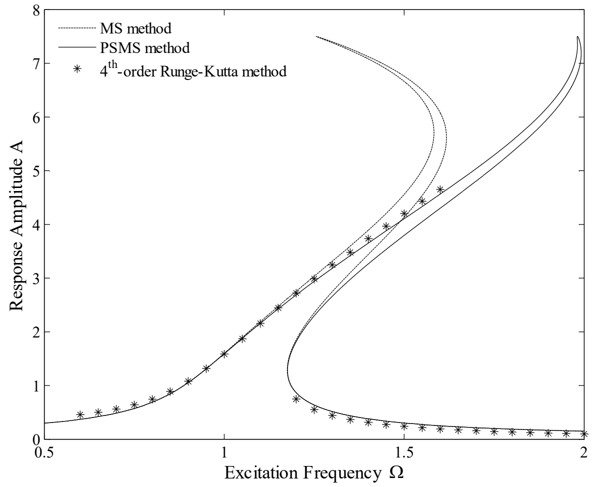

Figure 2. FRCs of oscillator 2 by the proposed method, the MS method and numerical simulation.

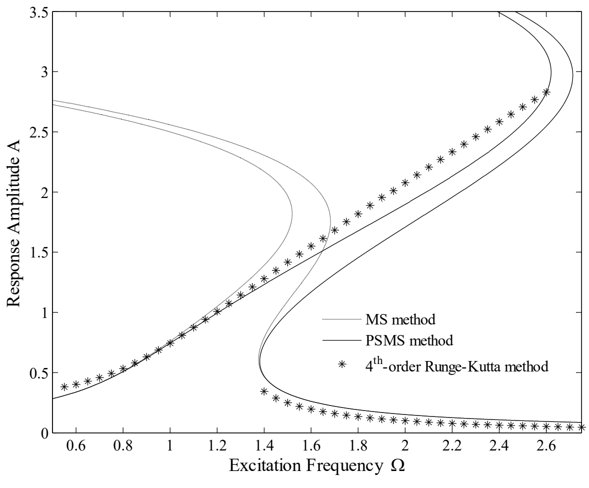

Figure 3. FRCs of oscillator 3 by the proposed method, the MS method and numerical simulation.

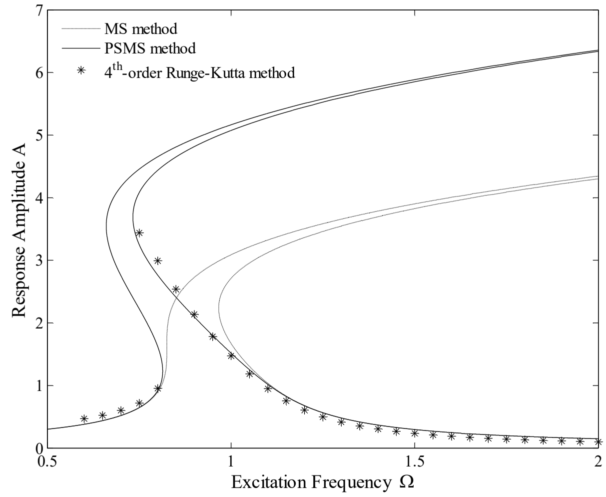

Figure 4. FRCs of oscillator 4 by the proposed method, the MS method and numerical simulation.

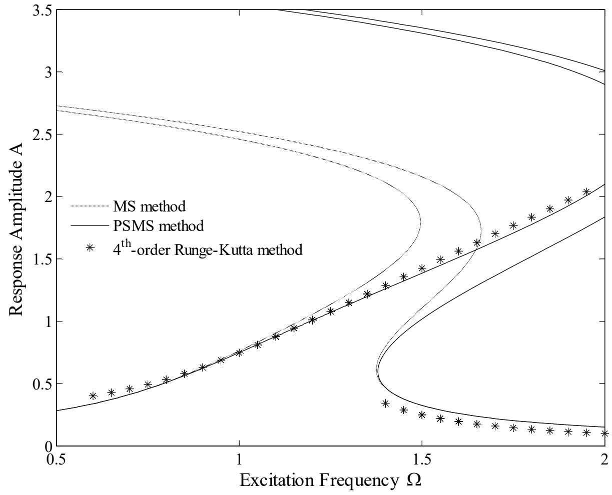

Figure 5. FRCs of oscillator 5 by the proposed method, the MS method and numerical simulation.

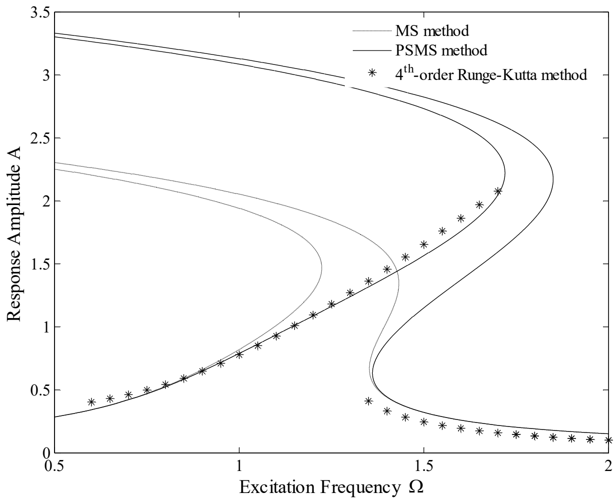

Figure 6. FRCs of oscillator 6 by the proposed method, the MS method and numerical simulation.