A novel singular element for analysis of stress-dimensional singular stress fields at circular crack fronts

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Abstract

Traditionally, fracture mechanics mainly considers two plane states, i.e., plane stress and plane strain states, these simple 2D cases make the equations for singular stress fields much easy to obtain by analytical mehtods [1]. Such essentially 2D theories have been very successful in dealing with fracture and fatigue on the macro-scale where the crack front can often be assumed to be a straight line. In fact, the crack front is usually curved, thereby, the three-dimensional character of the crack problem must be taken into account. A small number of fracture problems have close form analytic solutions due to the difficulty of mathematics [2-5]. In such cases, different numerical methods were presented to treat the curved cracks [6-10]. It is noticed that the finite element method (FEM) is one of the most widely used numerical methods for this problem, in which special elements were developed for stress intensity factor (SIF) evaluation. The shape functions with a $1/r^{1/2}$ singularity at the crack front are used to establish the singular quarter-point element, which make it mainly exists in the crack problems of homogeneous materials [10-12]. Some scholars constructed interpolation functions for special elements by using the leading-order terms of asymptotic solutions [13]. There exist three shortcomings in the special elements with built-in leading-order singularity and conventional elements: the first one is that the numerical results are still dependent on the special element's size; the second one is that these enriched elements usually need transition elements in order to improve the accuracy of the numerical results; thirdly, inter-element compatibility between the special and conventional elements can't be satisfied due to different number of degrees of freedom at nodes. In contrast, special elements with built-in series solutions of asymptotic displacement and stress fields is insensitive to the element sizes, therefore, highly refined meshes are not necessary near the crack tips, and the computational efficiency in the finite element analysis can be improved [14, 15]. However, a special element with built-in series solutions for 3D curved crack problems is still needed considering all of the existing special elements are for 2D problems. For this purpose, a novel super singular element containing a part of a circular crack front is developed herein based on the series eigen-solutions as well as Hellinger-Reissner variational principle.

Numerical eigen-solutions of 3-D displacement and stress fields solved from a one-dimensional finite element formulation for generalized plane strain problems [16] are resorted to herein. The assumed 3-D displacement and stress fields in the super hybrid singular element have satisfied the governing differential equations and compatibility equations in advance. As a result, the displacement and stress vectors can be written as

$$\boldsymbol{u}(\rho,\theta)=\boldsymbol{U}\boldsymbol{\beta} \tag{1}$$

$$\boldsymbol{\sigma}(\rho,\theta)=\boldsymbol{\Sigma}\boldsymbol{\beta} \tag{2}$$

in which $\boldsymbol{u}(\rho,\theta)$ and $\boldsymbol{\sigma}(\rho,\theta)$ are total asymptotic displacements and stresses near the circular crack front in the cylindrical coordinate system (see Fig. 1), respectively, $\boldsymbol{U}$ and $\boldsymbol{\Sigma}$ include displacement and stress variations, respectively, due to the existence of n-order stress singularities, and $\boldsymbol{\beta}$ contains unknown multiplicative coefficients to be determined.

In general, the multiplicative coefficients β_i in Eqs. (1) and (2) vary with the crack front for the specific case of a 3D crack, unlike the 2D case where they are assumed to be a constant value for a given geometry and loading. In order to establish a 3D circular crack front

isoparametric element, a new natural coordinate η is introduced to represent the position of a normal plane along the crack front, and the coefficients $\beta_i(\eta)$ are assumed to be a linear function of η in each element, as a result, the assumed displacement and stress components in Cartesian coordinate system can be written as

$$\boldsymbol{u}(x,y,z)=\boldsymbol{Z}_u\boldsymbol{u}(\rho,\theta)=\boldsymbol{Z}_u(1-\eta)/2\cdot\boldsymbol{U}\boldsymbol{\beta}_1+\boldsymbol{Z}_u(1+\eta)/2\cdot\boldsymbol{U}\boldsymbol{\beta}_2=\boldsymbol{Z}_u\boldsymbol{U}_\mathrm{c}\boldsymbol{\beta}_\mathrm{c} \tag{3}$$

$$\boldsymbol{\sigma}(x,y,z)=\boldsymbol{Z}_\sigma\boldsymbol{\sigma}(\rho,\theta)=\boldsymbol{Z}_\sigma(1-\eta)/2\cdot\boldsymbol{\Sigma}\boldsymbol{\beta}_1+\boldsymbol{Z}_\sigma(1+\eta)/2\cdot\boldsymbol{\Sigma}\boldsymbol{\beta}_2=\boldsymbol{Z}_\sigma\boldsymbol{\Sigma}_\mathrm{c}\boldsymbol{\beta}_\mathrm{c} \tag{4}$$

The coordinate system transformation matrices $\boldsymbol{Z}_u$ and $\boldsymbol{Z}_\sigma$ are from the cylindrical coordinate system to the Cartesian coordinate system.

Based on the hybrid-stress finite element method, the boundary displacements $\tilde{\boldsymbol{u}}(x,y,z)$ are assumed separately from $\boldsymbol{u}(x,y,z)$, and are expressed in terms of the nodal displacements in the displacement-base finite element methods as

$$\tilde{\boldsymbol{u}}(x,y,z)=\boldsymbol{N}\overline{\boldsymbol{\delta}} \tag{5}$$

where the matrix $\boldsymbol{N}$ represents the shape function matrix for a four-node quadrilateral element allowing for 3D displacements at a node, and the vector $\overline{\boldsymbol{\delta}}^\mathrm{T}=[q_x\ q_y\ q_z]$ includes the nodal displacements.

Substituting $\boldsymbol{u}(x,y,z)$, $\boldsymbol{\sigma}(x,y,z)$ and $\tilde{\boldsymbol{u}}(x,y,z)$ of Eqs. (3) - (5) into a hybrid functional established by the Hellinger-Reissner principle, we obtain the element stiffness matrix for the super crack front element as shown in Fig. 2.

Numerical benchmark examples are presented to demonstrate the accuracy and efficiency of the super crack front elements. The first validation problem involves a circular penny-shaped crack in a 3D solid. The second validation problem is a cracked bar under tension and torsion. As application, interfacial crack problem will also be considered. The current investigation will pay close attention to the singular stress field along the circular crack front and all possible fracture modes in 3D circular cracks.

Keywords: 3D fracture, singular stress field, super singular element, series solution

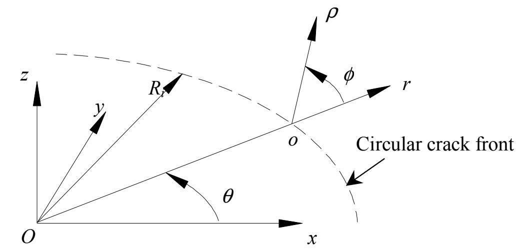

Fig. 1 Contact interface coordinate system

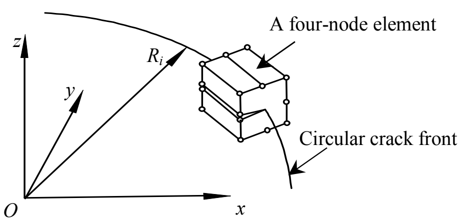

Fig. 2 A super singular element

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