On boundary-value problems of elasticity theory with mixed boundary conditions

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Abstract

We consider a class of mixed boundary value problems of elasticity theory for the junction of two rectangular horizontal semi-strips of the same width with different boundary conditions on their long sides. On the junction of semi-strips, the continuity conditions of solutions or the discontinuity of displacements and stresses can be known.

Keywords: Mixed boundary value problems of elasticity theory; junction of two semi-strips

Introduction

In the general case, the solutions of such problems are represented as series in Papkovich–Fadle eigenfunctions, in particular as series in trigonometric functions. The unknown expansion coefficients are determined from the conditions on the junction of semi-strips. However, since two complete and minimal systems of functions (for example, trigonometric ones) take part in the solutions, their union on the junction will not be minimal. Therefore, it is impossible to construct a system of functions that is biorthogonal to this union. Hence, it is impossible to find a closed form solution. The main idea is to form a minimal system of functions, then to construct biorthogonal systems of functions and determine the unknown expansion coefficients with their help.

Statement of the problem and its solving

Let us consider the horizontal strip $\{\Pi :|y|\le 1,|x|<\infty\}$ with the following boundary conditions:

$$\begin{gathered} u(x,\pm 1)=0,\ \sigma_y(x,\pm 1)=p^{+}(x),\ (x>0); \\ v(x,\pm 1)=\tau_{xy}(x,\pm 1)=0,\ \ (x<0), \end{gathered} \tag{1}$$

where u and v are displacements along the x- and y-axes respectively.

Suppose that $p(x)\in L_2(-\infty,\infty)$ is a certain continuation of $p^{+}(x)$ to the whole real axis. We will assume that the null boundary functions are continued by zero. The solutions in the left semi-strip $\{\Pi^{-}: x\le 0,|y|\le 1\}$ and in the right semi-strip $\{\Pi^{+}: x\ge 0,|y|\le 1\}$ can be represented in the form of Fourier series and integrals [1][2]. The corresponding formulas of the displacements and the stresses in the semi-strips $\Pi^{\mp}$ are as follows:

$$U^{-}(x,y)=\frac{1-\nu}{2}A_0 x-\frac{1+\nu}{2}\sum_{k=1}^{\infty}A_k\left(q_k^2+B_k q_k+B_k q_k^2 x\right)e^{q_k x}q_k\cos(q_k y),$$

$$V^{-}(x,y)=\sum_{k=1}^{\infty}\left\{\left(\frac{1+\nu}{2}A_k q_k^2+2B_k q_k\right)+\frac{1+\nu}{2}B_k q_k^2 x\right\}e^{q_k x}\sin(q_k y),$$

$$\sigma_x^-(x,y) = A_0 - \sum_{k=1}^{\infty}\left\{\left((1+\nu)A_k q_k^3 + 2B_k q_k^2\right) + (1+\nu)B_k q_k^3 x\right\} e^{q_k x}\cos(q_k y)\,, \qquad (2)$$

$$\sigma_y^-(x,y) = \nu A_0 + \sum_{k=1}^{\infty}\left\{\left((1+\nu)A_k q_k^3 + 2(2+\nu)B_k q_k^2\right) + (1+\nu)B_k q_k^3 x\right\} e^{q_k x}\cos(q_k y)\,,$$

$$\tau_{xy}^-(x,y) = \sum_{k=1}^{\infty}\left\{\left((1+\nu)A_k q_k^3 + (3+\nu)B_k q_k^2\right) + (1+\nu)B_k q_k^3 x\right\} e^{q_k x}\sin(q_k y)$$

and

$$U^+(x,y) = -\frac{1+\nu}{2}\sum_{k=1}^{\infty}(a_k p_k^2 + b_k p_k + b_k p_k^2 x)e^{p_k x}\cos(p_k y) + U^\circ(x,y)\,,$$

$$V^+(x,y) = \sum_{k=1}^{\infty}\left(\frac{1+\nu}{2}a_k p_k^2 + 2b_k p_k + \frac{1+\nu}{2}b_k p_k^2 x\right)e^{p_k x}\sin(p_k y) + V^\circ(x,y)\,,$$

$$\sigma_x^+(x,y) = -\sum_{k=1}^{\infty}\left\{(1+\nu)\,a_k p_k^3 + 2b_k p_k^2 + (1+\nu)b_k p_k^3 x\right\} e^{p_k x}\cos(p_k y) + \sigma_x^\circ(x,y)\,, \qquad (3)$$

$$\sigma_y^+(x,y) = \sum_{k=1}^{\infty}\left\{(1+\nu)a_k p_k^3 + 2(2+\nu)b_k p_k^2 + (1+\nu)b_k p_k^3 x\right\} e^{p_k x}\cos(p_k y) + \sigma_y^\circ(x,y)\,,$$

$$\tau_{xy}^+(x,y) = \sum_{k=1}^{\infty}\left\{(1+\nu)a_k p_k^3 + (3+\nu)b_k p_k^2 + (1+\nu)b_k p_k^3 x\right\} e^{p_k x}\sin(p_k y) + \tau_{xy}^\circ(x,y)\,.$$

In formulas (2) and (3) the following notations are introduced: G is the shear modulus, ν is the Poisson ratio, $q_k = k\pi$, $p_k = -(2k-1)\pi/2$, $U(x,y) = Gu(x,y)$ and $V(x,y) = Gv(x,y)$. The superscript $\circ$ indicates the quantities corresponding to the solution in terms of Fourier integrals for the infinite strip under the boundary conditions $U(x,\pm 1) = 0$, $\sigma_y(x,\pm 1) = p(x)$. This solution can be found easily, for example:

$$U^\circ(x,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1+\nu}{4}(\sin t\cos ty - y\cos t\sin ty)\frac{e^{tx}}{\cos^2 t}\Im[p](t)dt\,. \qquad (4)$$

$\Im[p](t)$ is the Fourier transform of the function $p(x)$.

Let $p^+(x) = p = const$ and $p(x) = p$. In this case the solution written in terms of Fourier integrals has the form

$$U^\circ(x,y) = 0,\; V^\circ(x,y) = \frac{1}{2}(1-\nu)py,\; \sigma_x^\circ(x.y) = p\nu,\; \tau_{xy}^\circ(x,y) = 0\,. \qquad (5)$$

The required coefficients A_k, B_k and a_k, b_k must be found from the continuity conditions of the displacements $U^\pm(0,y)$ and $V^\pm(0,y)$ and the stresses $\sigma_x^\pm(0,y)$ and $\tau_{xy}^\pm(0,y)$ on the junction of the semi-strips. As a result, we obtain four functional equations containing the four complete minimal systems of functions $\{1,\cos q_k y\}\cup\{\cos p_k y\}$ and $\{\sin q_k y\}\cup\{\sin p_k y\}$ $(k \ge 1)$. However, these unions are not minimal. It follows that the unknown coefficients cannot exactly be found from these equations. We must first eliminate the unnecessary functions from these equations. In order to do that, we consider the two functions analytic in the semi-strips $\Pi^\pm$:

$$\Phi^{\pm}(x,y)=\frac{i}{2}\left(2(1+\nu)\frac{dV^{\pm}(x,y)}{dy}-(1-\nu)\sigma_x^{\pm}(x,y)\right)-\tau_{xy}^{\pm}(x,y),$$
$$\Psi^{\pm}(x,y)=(1+\nu)\frac{dU^{\pm}(x,y)}{dy}-\frac{3+\nu}{2}\tau_x^{\pm}(x,y)+i\left(\nu\sigma_x^{\pm}(x,y)+2(1+\nu)\frac{dV^{\pm}(x,y)}{dy}\right). \tag{6}$$

By expanding the equations $\Phi^{+}(0,y)=\Phi^{-}(0,y)$ and $\Psi^{+}(0,y)=\Psi^{-}(0,y)$, we obtain the system of two functional equations

$$-(1-\nu)\frac{A_0}{2}+\sum_{k=1}^{\infty}[(1+\nu)A_k q_k^3+(3+\nu)B_k q_k^2]e^{iq_k y}-$$
$$-\sum_{k=1}^{\infty}[(1+\nu)a_k p_k^3+(3+\nu)]b_k p_k^2 e^{iq_k y}=\frac{\Phi^{\circ}(y)}{i},$$
$$mA_0+\sum_{k=1}^{\infty}[(1+\nu)A_k q_k^3+(4+2\nu)B_k q_k^2]e^{iq_k y}-$$
$$-\sum_{k=1}^{\infty}[(1+\nu)a_k p_k^3+(4+2\nu)b_k p_k^2]e^{ip_k y}=\frac{\Psi^{\circ}(y)}{i}. \tag{7}$$

The functions $\Phi^{\circ}(y)$ and $\Psi^{\circ}(y)$ are determined according to (6) for the variables indicated by the degree superscript. In accordance with (5), we have

$$\Phi^{\circ}(y)=(1-\nu)pi/2,\ \Psi^{\circ}(y)=pi.$$

Now we introduce the new notations:

$$\omega_1=q_1,\ \omega_2=p_1,\omega_3=q_2,\quad \omega_4=p_2,...,$$
$$D_1=A_1,D_2=-a_1,D_3=A_3,D_4=-a_2,...,$$
$$C_1=B_1,C_2=-b_1,C_3=B_3,C_4=-b_2,.... \tag{8}$$

Theorem 1. The function system $\{e^{i\omega_k y}\}_{k=1}^{\infty}$ is complete and minimal in $L_2(-\infty,\infty)$.

Theorem 2. There exist a unique function system $\{\psi_k(y)\}_{k=1}^{\infty}$ biorthogonal to $\{e^{i\omega_k y}\}_{k=1}^{\infty}$. These functions are given by the formulas

$$\psi_k(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{L(\omega)\exp(-i\omega y)d\omega}{(\omega-\omega_k)L'(\omega_k)}\quad (|y|\le 1) \tag{9}$$

We multiply Eqs. (7) by $\psi_k(y)$ and integrate the result over $[-1,1]$ for each $k\ge 1$, thus obtaining the system of two algebraic equations for the unknowns C_k and D_k. Solving it and writing the result in the original notations (8), we have

$$A_k=\frac{(1-\nu)p}{(1+\nu)p_k^4 L'(q_k)},\ B_k=\frac{(1-\nu)p}{2q_k^3 L'(q_k)},$$
$$a_k=\frac{-(1-\nu)p}{(1+\nu)p_k^4 L'(p_k)},\ b_k=\frac{(1-\nu)p}{2p_k^3 L'(p_k)}. \tag{10}$$

By introducing the function

$$\varphi(n)=\frac{(-1)^{n-1}(n-1)!}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)},$$

we can write out the final formulas for the stresses:

$$\frac{\sigma_x^-(x,y)}{p}=\nu+\frac{x}{2}(1-\nu^2)\sum_{n=1}^{\infty}\frac{\exp(n\pi x)}{\varphi(n)}\cos(n\pi)\,,$$
$$\frac{\sigma_y^-(x,y)}{p}=\nu^2-(1-\nu^2)\sum_{n=1}^{\infty}\left\{\frac{1}{n\pi}+\frac{x}{2}\right\}\frac{\exp(n\pi x)}{\varphi(n)}\cos(n\pi y)\,, \qquad (11)$$
$$\frac{\tau_{xy}^-(x,y)}{p}=-\frac{(1-\nu^2)}{2}\sum_{n=1}^{\infty}\left\{\frac{1}{n\pi}+x\right\}\frac{\exp(n\pi x)}{\varphi(n)}\sin(n\pi y),$$

$$\frac{\sigma_x^+(x,y)}{p}=\nu+\frac{x}{2}(1-\nu^2)\sum_{n=1}^{\infty}\frac{1}{\varphi(n)}\exp\left(-\frac{2n-1}{2}\pi x\right)\cos\frac{2n-1}{2}\pi y\,,$$
$$\frac{\sigma_y^+(x,y)}{p}=1+(1-\nu^2)\sum_{n=1}^{\infty}\left\{\frac{2}{(2n-1)\pi}-\frac{x}{2}\right\}\frac{1}{\varphi(n)}\exp\left(-\frac{2n-1}{2}\pi x\right)\cos\frac{2n-1}{2}\pi y \qquad (12)$$
$$\frac{\tau_{xy}^+(x,y)}{p}=-\frac{(1-\nu^2)}{2}\sum_{n=1}^{\infty}\left\{\frac{2}{(2n-1)\pi}-x\right\}\frac{1}{\varphi(n)}\exp\left(-\frac{2n-1}{2}\pi x\right)\sin\frac{2n-1}{2}\pi y\,.$$

As an illustration, Fig. 1 shows the distribution of the stresses $\sigma_x^{\pm}(\pm 0.05,y)$.

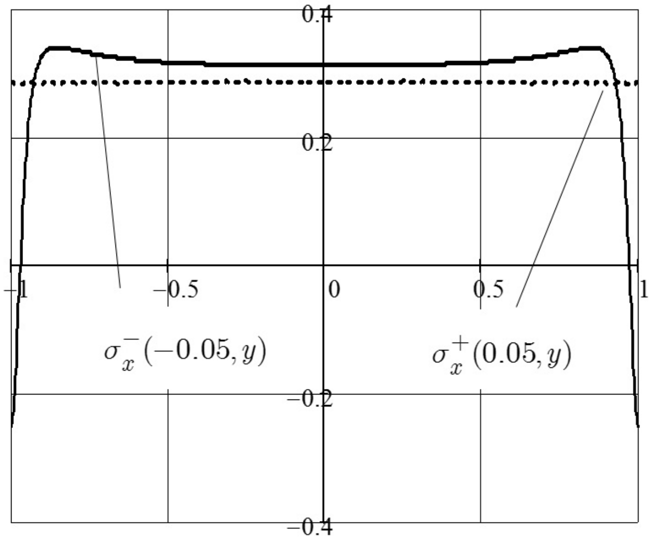

Figure 1. Distribution of the stresses $\sigma_x^{\pm}(\pm 0.05,y)$

Conclusions

It can be shown that the biorthogonal functions have a singularity of the type $\left(1\pm y\right)^{-1/2}$, therefore, the stresses at these points will also have the same singularity.

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References

[1] Kovalenko, M. D., Menshova, I. V. and Kerzhaev, A. P. (2015) Method of initial functions and the integral Fourier transform in the boundary value problem of elasticity theory for an infinite strip (I), *Mechanics of Composite Materials and Structures* **21**(1), 47–64 [in Russian].

[2] Kovalenko, M. D., Menshova, I. V. and Kerzhaev, A. P. (2015) Method of initial functions and the integral Fourier transform in the boundary value problem of elasticity theory for an infinite strip (II), *Mechanics of Composite Materials and Structures* **21**(2), 237–250 [in Russian].