Extrapolation Methods for Computing Supersingular Integral on a Circle

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Abstract

The trapezoidal rule for the computation of supersingular integrals on circle is discussed, and the asymptotic expansion of error function is obtained. A series to approach the singular point is constructed. The extrapolation algorithm is presented and the convergence rate is proved. Some numerical results are also reported to confirm the theoretical results and show the efficiency of the algorithms.

Keywords: Supersingular integral; Extrapolation method; Composite trapezoidal rule; Posteriori estimate

1.Introduction

Accurate calculation of boundary element methods(BEM) arising in boundary integral equations has been a subject of intensive research in recent years. The formulation of certain classes of boundary value problems in terms of supersingular integral equations:

$$\fint_a^b \frac{f(x)}{(x-s)^{p+1}}dx = g(s) \quad s\in(a,b), p=1,2 \tag{1}$$

have drawn lots of interests. In the literature different definitions of singular integrals are found which can be shown to be the same. We mention the following one

$$\fint_a^b \frac{f(x)}{(x-s)^{p+1}}dx = \lim_{\varepsilon\to 0}\left\{\left(\int_a^{s-\varepsilon}+\int_{s+\varepsilon}^b\right)\frac{f(x)}{(x-s)^{p+1}}dx - \frac{2f^{(p-1)}(s)}{\varepsilon}\right\},$$
$$s\in(a,b), \quad p=1,2. \tag{2}$$

where $\fint_a^b$ denotes a supersingualr integral and s the singular point. Here the supersingular integral is one order higher singularity than hypersingular integral.

One of the major problems arising from boundary element methods is how to evaluate such supersingular integral efficiently. For the case $p=1$, numerous work has been devoted to developing efficient quadrature formulas for hypersingular integral such as the Gaussian method [7,8], the Newton-Cotes method [15,22-25], the transformation method [3,5] and some other methods [4,10,19]Because of the high-order singularity of the kernels, the rules for Hadamard finite-part integrals (including hypersingular and supersingular integrals) are less accurate than their counterparts for Riemann integrals. Newton-Cotes rules for evaluating hypersingular integrals were firstly suggested by Linz [15]. The superconvergence phenomenon of trapezoidal rule and Simpson's rule for hypersingular integrals was found in [22,24], which showed that one order higher superconvergence rate than that in general case if the singular point is located at some a prior known point. Then, the superconvergence for arbitrary degree Newton-Cotes rules of hypersingular integrals were studied in [22,23] and the superconvergence rate was $O(h^{k+1})$. Recently, Newton-Cotes rules and the corresponding superconvergence for evaluating hypersingular integrals on a circle were discussed in [29,30]. Integrals with kernels beyond hypersingular have not been extensively studied with $p=2$. Du [4] studied the composite Simpson's rule and showed the optimal global convergence rate is $O(h)$. Then, Wu & Sun [22] studied the superconvergence of trapezoidal rule, the $O(h^2)$ superconvergence rate was obtained when the singular point is located at the middle point of each subinterval far away from two endpoints. Recently, Zhang et.al [28] discussed the superconvergence phenomenon of the composite Simpson's rule and also the $O(h^2)$ rate was obtained for those superconvergence points far away from the endpoints.

In this paper, we consider the supersingular integral defined in a circle which have been paid less attention to it.

$$\begin{aligned} I(f,s): &= \ne\!\!\!\!\!\!\int_c^{c+2\pi} \frac{\cos\frac{x-s}{2} f(x)}{\sin^3\frac{x-s}{2}} dx \\ &= \lim_{\epsilon\to 0}\left\{ \int_c^{s-\epsilon} \frac{f(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}\, dx + \int_{s+\epsilon}^{c+2\pi} \frac{f(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}\, dx - \frac{2\epsilon f'(s)}{\sin^2\frac{\epsilon}{2}} \right\}. \end{aligned} \tag{3}$$

To our knowledge, maybe the reference [12-14] have entire on the subject, where the superconvegence rate of Simpson rule and trapezoidal rule to compute the supersingular integral have been considered. When the singular point coincides with some priori known point, the convergence rate of the trapezoidal rule is higher than the global one which is considered as the superconvergence phenomenon. Then the error functional of density function is derived and the superconvergence phenomenon of composite trapezoidal rule occurs at certain local coordinate of each subinterval which is different from the case supersingular defined on the interval.

Extrapolation methods as an accelerating convergence technique has been applied to many fields in computational mathematics [16,21]. The most famous one is Richardson extrapolation with the error function as
$T(h) - a_0 = a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots$,
where $T(0) = a_0$ and a_i are constant independent of *h*. Then in the paper of Li [11] et.al, the trapezoidal rule for computation hypersingular integral on interval by extrapolation methods was given.

Before presented the extrapolation methods to compute the supersingular integral in a circle, we firstly give notation as below. Let

$$\psi_{ik}(t) = \begin{cases} \ne\!\!\!\!\!\!\int_{-1}^{1} \frac{\mathcal{M}_{ik}(\tau,t)\cos\frac{\tau-t}{2}}{(k-i)!\sin^3\frac{\tau-t}{2}} d\tau, & |t|<1, \\ \int_{-1}^{1} \frac{\mathcal{M}_{ik}(\tau,t)\cos\frac{\tau-t}{2}}{(k-i)!\sin^3\frac{\tau-t}{2}} d\tau, & |t|>1, \end{cases} \tag{4}$$

where

$$\mathcal{M}_{ik}(\tau,t) = (\tau^2-1)(\tau-t)^{k-i}[(\tau+1)^i - (\tau-1)^i]. \tag{5}$$

Let $J := (-\infty,-1)\bigcup(-1,1)\bigcup(1,\infty)$ and the operator $W: C(J) \to C(-1,1)$ be defined by

$$T_{ik}(\tau) := \psi_{ik}(\tau) + \sum_{j=1}^{\infty} \psi_{ik}(2j+\tau) + \sum_{j=1}^{\infty} \psi_{ik}(-2j+\tau), \quad k = i, i+1. \tag{6}$$

In this paper, based on the asymptotic error expansion of the composite trapezoidal rule for the computation of supersingular integrals. We firstly obtain the asymptotic error expansion as follow

$$E_n(f,s) = \sum_{i=1}^{l-2} \frac{h^{i-1}}{2^{i-1}} f^{(i+1)}(s) a_i(\tau) + O(h^{l-2}), \tag{7}$$

where $a_i(\tau)$ are functions independent of h, τ the local coordinate of the singular point. Then an extrapolation algorithm is presented to compute the supersingular integral. For a given τ a series of s_j is selected to approximate the singular point *s* with local coordinate equal the superconvergence point accompanied by the refinement of the meshes. Moreover, by means of the extrapolation technique, we not only obtain an approximation with higher order accuracy, but also get a posteriori error estimate.

The rest of this paper is organized as follows. In Sect.2, after introducing some basic formulas of the general (composite) trapezoidal rule, some notations and preliminaries, we present our main result. In Sect.3 the proof is completed. In Sect.4, extrapolation algorithm is presented and the convergence rate is proved. Finally, several numerical examples are given to validate our analysis.

2.Main result

Let $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = c + 2\pi$ be a uniform partition of the interval $[c, c+2\pi]$ with mesh size $h = 2\pi/n$ and $f_L(x)$ the piecewise linear interpolant for $f(x)$:

$$f_L(x) = \frac{x - x_{j-1}}{h} f(x_j) + \frac{x_j - x}{h} f(x_{j-1}), \quad x \in [x_{j-1}, x_j], \quad 1 \le j \le n, \tag{8}$$

and a linear transformation

$$\text{x=}_j(\tau) := (\tau + 1)(x_j - x_{j-1})/2 + x_{j-1}, \quad \tau \in [-1, 1], \tag{9}$$

maps the reference element $[-1, 1]$ onto the subinterval $[x_{j-1}, x_j]$. Replacing $f(x)$ in (3) with $f_L(x)$ gives the composite trapezoidal rule:

$$I_n(f, s) := \fint_c^{c+2\pi} \frac{\cos\frac{x-s}{2} f_L(x)}{\sin^3\frac{x-s}{2}} dx = \sum_{j=0}^{n} \omega_j(s) f(x_j) = I(f, s) - E_n(f, s), \tag{10}$$

where $\omega_j(s)$ denotes the Cotes coefficients, see [13] and $E_n(f, s)$ the error functional.

We define

$$F_i(\tau) = (\tau - 1)(\tau + 1)[(\tau + 1)^i - (\tau - 1)^i] \tag{11}$$

and

$$\phi_{i,i+2}(t) = \begin{cases} -\dfrac{1}{2} \displaystyle\fint_{-1}^{1} \frac{F_i(\tau)}{\tau - t} d\tau, & |t| < 1, \\ -\dfrac{1}{2} \displaystyle\int_{-1}^{1} \frac{F_i(\tau)}{\tau - t} d\tau, & |t| > 1. \end{cases} \tag{12}$$

Suppose $F_i(\tau)$ is replaced by the Legendre polynomial, then $\phi_{i,i+2}(t)$ defines the Legendre function of the second kind \cite{candrews1992}. Let

$$\phi_{i,i+1}(t) = \phi'_{i,i+2}(x), \tag{13}$$

$$\phi_{ii}(t) = 2\phi'_{i,i+1}(t) = 2\phi''_{i,i+2}(t) \tag{14}$$

and

$$\phi_{ik}(t) = -\frac{1}{2} \int_{-1}^{1} \frac{F_i(\tau)(\tau - t)^{k-i-3}}{(k-i)!} d\tau, \quad k > i + 2. \tag{15}$$

Before presenting the main results, we firstly define $K_s(x)$ as follows

$$K_s(x) = \begin{cases} \dfrac{(x-s)^3 \cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} & x \ne s, \\ 8, & x = s. \end{cases} \tag{16}$$

We present our main results below.

Theorem 1 Assume $f(x) \in C^l[a, b], \quad l \ge 3$. For the trapezoidal rule $I_n(f; s)$ defined in (10), there exists a positive constant *C*, independent of *h* and *s* and functions $a_i(\tau)$, independent of *h*, such that

$$E_n(f, s) = \sum_{i=1}^{l-2} \frac{h^{i-1}}{2^{i-1}} f^{(i+1)}(s) a_i(\tau) + R_n(s), \tag{17}$$

where $s = x_{m-1} + (1 + \tau)h/2, \quad m = 1, 2, \cdots, n$ and

$$|R_n(s)| \le C \max_{x \in (c, c+2\pi)} \{K_s(x)\}(|\ln h| + \gamma^{-2}(\tau))h^{l-2} \tag{18}$$

and

$$\gamma(\tau) = \min_{0\le j\le n} \frac{|s-x_j|}{h} = \frac{1-|\tau|}{2} \tag{19}$$

3.Proof of the Theorem 1

In the following analysis, *C* will denote a generic positive constant which is independent of *h* and *s*. Let P_l and Q_l denote the Legendre polynomial [1] of degree *l* and the associated Legendre function of the second kind, respectively.
Define

$$M_{ik}^{j}(x,s) \quad = F_i^j(x)(x-s)^{k-i}, \quad k\ge i, \tag{20}$$

where

$$F_i^j(x) = (x-x_{j-1})(x-x_j)[(x-x_{j-1})^i - (x-x_j)^i]. \tag{21}$$

By (9), we have

$$M_{ik}^{j}(x,s) \quad = \frac{h^{k+3}}{2^{k+3}} F_i(\tau)(\tau-c_j)^{k-i}, \tag{22}$$

where $F_i(\tau)$ is defined by (21) and

$$M_{ik}(\tau, c_j) = (\tau^2-1)(\tau-c_j)^{k-i}[(\tau+1)^i - (\tau-1)^i] \tag{23}$$

and

$$c_j = \frac{2}{h}(s-x_{j-1}) - 1. \tag{24}$$

Lemma 3.1 Let $K_s(x)$ be defined in (16). For $x\in(x_{j-1},x_j)$, by linear transformation (9), we have

$$K_s(x) = K_{c_j}(\tau), \quad \tau\in(-1,1) \tag{25}$$

where

$$K_{c_j}(\tau) = 8 + 8\sum_{l=1}^{\infty}\frac{(\tau-c_j)^3}{(\tau-c_j-2ln)^3} + 8\sum_{l=1}^{\infty}\frac{(\tau-c_j)^3}{(\tau-c_j+2ln)^3} \tag{26}$$

And c_j is defined as (24).
Proof:By the identity in [1]

$$\frac{\pi^3\cos\pi t}{\sin^3\pi t} = \sum_{l=-\infty}^{\infty}\frac{1}{(t+l)^3}, \tag{27}$$

then we get

$$\frac{\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} = \frac{8}{(x-s)^3} + 8\sum_{l=1}^{\infty}\frac{1}{(x-s-2l\pi)^3} + 8\sum_{l=1}^{\infty}\frac{1}{(x-s+2l\pi)^3} \tag{28}$$

and

$$\begin{aligned} K_s(x) &= \frac{(x-s)^3\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} = 8 + 8\sum_{l=1}^{\infty}\frac{(\tau-c_j)^3}{(\tau-c_j-4l\pi/h)^3} + 8\sum_{l=1}^{\infty}\frac{(\tau-c_j)^3}{(\tau-c_j+4l\pi/h)^3} \\ &= 8 + 8\sum_{l=1}^{\infty}\frac{(\tau-c_j)^3}{(\tau-c_j-2ln)^3} + 8\sum_{l=1}^{\infty}\frac{(\tau-c_j)^3}{(\tau-c_j+2ln)^3} \\ &= K_{c_j}(\tau), \end{aligned}$$

which completes the proof.
Lemma 3.2 Assume $s\in(x_{m-1},x_m)$ for some *m* and let $c_j(1\le j\le n)$ be given by (24). Then we have

$$\phi_{ik}(c_j)=\begin{cases} -\dfrac{2^{k-1}}{h^{k-1}}\displaystyle\neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m}\frac{\cos\frac{x-s}{2}M_{ik}^m(x,s)}{\sin^3\frac{x-s}{2}}dx, & j=m, \\ -\dfrac{2^{k-1}}{h^{k-1}}\displaystyle\int_{x_{j-1}}^{x_j}\frac{\cos\frac{x-s}{2}M_{ik}^j(x,s)}{\sin^3\frac{x-s}{2}}dx, & j\neq m. \end{cases} \tag{29}$$

Proof: If $j=m$, by the definition of (3) and noting $k=i$, we have

$$\begin{aligned} \neq\!\!\!\!\!\!\int_{x_{j-1}}^{x_j}\frac{\cos\frac{x-s}{2}M_{ik}^j(x,s)}{\sin^3\frac{x-s}{2}}dx &= \neq\!\!\!\!\!\!\int_{x_{j-1}}^{x_j}\frac{M_{ik}^j(x,s)K_s(x)}{(x-s)^3}dx \\ &= \neq\!\!\!\!\!\!\int_{x_{j-1}}^{x_j}\frac{F_i^j(x)K_s(x)}{(x-s)^3}dx \\ &= \lim_{\epsilon\to 0}\left\{\left(\int_{x_{j-1}}^{s-\epsilon}+\int_{s+\epsilon}^{x_j}\right)\frac{F_i^j(x)K_s(x)}{(x-s)^3}dx-\frac{2F_i^j(s)K_s(s)}{\epsilon}\right\} \\ &= \frac{h^{k-1}}{2^{k-1}}\lim_{\epsilon\to 0}\left\{\left(\int_{-1}^{c_m-\frac{2\epsilon}{h}}+\int_{c_m+\frac{2\epsilon}{h}}^{1}\right)\frac{F_i(\tau)K_s(x)}{(\tau-c_m)^3}d\tau-\frac{hF_i(c_m)K_s(c_m)}{\epsilon}\right\} \\ &= \frac{h^{k-1}}{2^{k-1}}\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{F_i(\tau)K_s(c_m)}{(\tau-c_m)^3}d\tau \\ &= -\frac{h^{k-1}}{2^{k-1}}\psi_{ii}(c_m). \end{aligned}$$

The case $j\neq m$ can be proved by applying the same approach to the correspondent Riemann integral.

Lemma 3.3 Under the assumption of Lemma3.2, for $k=i+1$, there holds that

$$\phi_{ik}(c_j)=\begin{cases} -\dfrac{2^{k-1}}{h^{k}}\displaystyle\neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m}\frac{\cos\frac{x-s}{2}M_{ik}^m(x,s)}{\sin^3\frac{x-s}{2}}dx, & j=m, \\ -\dfrac{2^{k-1}}{h^{k}}\displaystyle\int_{x_{j-1}}^{x_j}\frac{\cos\frac{x-s}{2}M_{ik}^j(x,s)}{\sin^3\frac{x-s}{2}}dx, & j\neq m. \end{cases} \tag{30}$$

Proof:By the definition of (3), we have:

$$\begin{aligned} \neq\!\!\!\!\!\!\int_{x_{j-1}}^{x_j}\frac{\cos\frac{x-s}{2}M_{ik}^j(x,s)}{\sin^3\frac{x-s}{2}}dx &= \neq\!\!\!\!\!\!\int_{x_{j-1}}^{x_j}\frac{M_{ik}^j(x,s)K_s(x)}{(x-s)^3}dx \\ &= \neq\!\!\!\!\!\!\int_{x_{j-1}}^{x_j}\frac{F_i^j(x)K_s(x)}{(x-s)^2}dx \\ &= \lim_{\varepsilon\to 0}\left\{\int_{x_{j-1}}^{s-\varepsilon}\frac{F_i^j(x)K_s(x)}{(x-s)^2}dx+\int_{s+\varepsilon}^{x_j}\frac{F_i^j(x)K_s(x)}{(x-s)^2}dx-\frac{2F_i^j(s)K_s(s)}{\varepsilon}\right\} \\ &= \frac{h^k}{2^k}\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{c_m-\frac{2\varepsilon}{h}}+\int_{c_m+\frac{2\varepsilon}{h}}^{1}\right)\frac{F_i(\tau)K_s(\tau)}{(\tau-c_m)^2}d\tau-\frac{hF_i(c_m)K_s(c_m)}{\varepsilon}\right\} \\ &= \frac{h^k}{2^k}\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{F_i(\tau)K_s(\tau)}{(\tau-c_m)^2}d\tau \\ &= -\frac{h^k}{2^{k-1}}\psi_{i,i+1}(c_m). \end{aligned}$$

The first identity in (30) is then verified. The second identity can be obtained by applying the approach to the correspondent Riemann integral. The proof is completed.

Lemma 3.4 Under the assumption of Lemma3.3, for $k=i+2$, there holds that

$$\phi_{ik}(c_j)=\begin{cases} -\dfrac{2^{k}}{h^{k+1}}\displaystyle\neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m}\frac{\cos\frac{x-s}{2}M_{ik}^m(x,s)}{\sin^3\frac{x-s}{2}}dx, & j=m, \\ -\dfrac{2^{k}}{h^{k+1}}\displaystyle\int_{x_{j-1}}^{x_j}\frac{\cos\frac{x-s}{2}M_{ik}^j(x,s)}{\sin^3\frac{x-s}{2}}dx, & j\neq m. \end{cases} \tag{31}$$

Proof:By the definition of (3), we have:

$$\begin{aligned}
\fint_{x_{j-1}}^{x_j} \frac{\cos\frac{x-s}{2} M_{ik}^j(x,s)}{\sin^3\frac{x-s}{2}}dx &= \fint_{x_{j-1}}^{x_j} \frac{M_{ik}^j(x,s)K_s(x)}{(x-s)^3}dx \\
&= \fint_{x_{j-1}}^{x_j} \frac{F_i^j(x)K_s(x)}{x-s}dx \\
&= \lim_{\varepsilon\to 0}\left\{\int_{x_{j-1}}^{s-\varepsilon} \frac{F_i^j(x)K_s(x)}{x-s}dx + \int_{s+\varepsilon}^{x_j} \frac{F_i^j(x)K_s(x)}{x-s}dx\right\} \\
&= \frac{h^{k+1}}{2^{k+1}}\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{c_m-\varepsilon} + \int_{c_m+\varepsilon}^{1}\right)\frac{F_i(\tau)K_s(\tau)}{\tau-c_m}d\tau\right\} \\
&= \frac{h^{k+1}}{2^{k+1}}\fint_{-1}^{1} \frac{F_i(\tau)K_s(\tau)}{\tau-c_m}d\tau \\
&= -\frac{h^{k+1}}{2^k}\psi_{i,i+2}(c_m)
\end{aligned}$$

The first identity in (31) is then verified. The second identity can be obtained by applying the approach to the correspondent Riemann integral. The proof is completed.

Lemma 3.5 Under the assumption of Lemma3.3 and for $k > i+2$, there holds that

$$\phi_{ik}(c_j) = -\frac{2^{k+1}}{h^{k+2}}\int_{x_{j-1}}^{x_j} \frac{M_{ik}^j(x,s)}{(k-i)!}dx. \tag{32}$$

The proof of this lemma can be obtained in a way similarly to that of Lemma 3.3 or Lemma 3.4.

Lemma 3.6 Suppose $f(x) \in C^l[c, c+2\pi], l \geq 3$. If $s \neq x_j$, for any $j = 1, 2, \cdots, n$, then there holds

$$\begin{aligned}
f(x) - f_L(x) &= \sum_{i=1}^{l-1}\sum_{k=i}^{l-1} \frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!}\frac{M_{ik}^j(x,s)}{(k-i)!} \\
&+ \sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\frac{f^{(l)}(\xi_{ij}) - f^{(l)}(s)}{(l-i-1)!}M_{i,l-1}^j(x,s) \\
&+ \frac{(-1)^l}{hl!}\tilde{M}_l^j(x), \xi_{ij} \in (x_{j-1}, x_j),
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
\tilde{M}_l^j(x) &= (x-x_{j-1})(x-t_j)\left[f^{(l)}(\eta_j)(x-x_{j-1})^{l-1} - f^{(l)}(\zeta_j)(x-x_j)^{l-1}\right] \\
&- f^{(l)}(s)M_{l-1,l-1}^j(x,s), \eta_j, \zeta_j \in (x_{j-1}, x_j).
\end{aligned}$$

Define

$$\mathcal{H}_m(x) = f(x) - f_L(x) - \sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!}\frac{M_{ik}^m(x,s)}{(k-i)!}, \quad x \in (x_{m-1}, x_m). \tag{34}$$

Lemma 3.7 Under the same assumptions of Theorem 1, for $\mathcal{H}_m(x)$ in (34), there holds that

$$\left|\fint_{x_{m-1}}^{x_m}\frac{\mathcal{H}_m(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right| \leq C\max_{x\in(x_{j-1},x_j)}\{K_s(x)\}\gamma^{-2}(\tau)h^2. \tag{35}$$

where $\gamma(\tau)$ is defined in (19).

Proof. By the definition of $\mathcal{H}_m(x)$, we have

$$|\mathcal{H}_m^{(i)}(x)| \leq Ch^{l-i}, i = 0, 1, 2. \tag{36}$$

As we known

$$\neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m} \frac{\mathcal{H}_m(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx = 8 \neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m} \frac{\mathcal{H}_m(x)}{(x-s)^3}dx + \neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m} \frac{\mathcal{H}_m(x)[\kappa_s(x)-8]}{(x-s)^3}dx. \quad (36)$$

From the identity

$$\begin{aligned} \neq\!\!\!\!\!\!\int_a^b \frac{f(x)}{(x-s)^3}dx = & \frac{f(s)}{2}\left[\frac{1}{(a-s)^2}-\frac{1}{(b-s)^2}\right] \\ & -\frac{(b-a)f'(s)}{(b-s)(s-a)} + \frac{f''(s)}{2}\ln\frac{b-s}{s-a} \\ & + \int_a^b \frac{f(x)-f(s)-f'(s)(x-s)-f''(s)(x-s)^2/2}{(x-s)^3}dx, \end{aligned} \quad (37)$$

we have

$$\begin{aligned} \neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m} \frac{\mathcal{H}_m(x)}{(x-s)^3}dx = & \frac{\mathcal{H}_m(s)}{2}\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right] \\ & -\frac{h\mathcal{H}_m'(s)}{(x_m-s)(s-x_{m-1})} \\ & +\frac{\mathcal{H}_m''(s)}{2}\ln\frac{x_m-s}{s-x_{m-1}} + \int_{x_{m-1}}^{x_m}\frac{\mathcal{H}_m'''(\theta(x))}{6}dx \end{aligned} \quad (38)$$

where $\theta(x) \in (x_{m-1}, x_m)$. Since

$$\begin{aligned} & \left|\frac{\mathcal{H}_m(s)}{2}\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right| \\ & = \left|\frac{\mathcal{H}_m(s)-\mathcal{H}_m(x_{m-1})}{2}\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right| \\ & = \left|\frac{\mathcal{H}_m'(\xi_{m-1})(s-x_{m-1})}{2}\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right| \\ & \le C\gamma^{-1}(\tau)h^{l-2}, \end{aligned} \quad (39)$$

where $\xi_{m-1} \in (x_{m-1}, x_m)$ and we have used $\mathcal{H}_m(x_{m-1}) = 0$.

$$\left|\frac{h\mathcal{H}_m'(s)}{(x_m-s)(s-x_{m-1})}\right| \le C\gamma^{-1}(\tau)h^{l-2},$$

$$\left|\frac{\mathcal{H}_m''(s)}{2}\ln\frac{x_m-s}{s-x_{m-1}}\right| \le C[|\ln\gamma(\tau)|+|\ln h|]h^{l-2}$$

and

$$\left|\int_{x_{m-1}}^{x_m}\frac{\mathcal{H}_m'''(\theta(x))}{6}dx\right| \le Ch^{l-2}.$$

As for the second term,

$$\begin{aligned} & \left|\neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m}\frac{\mathcal{H}_m(x)[\kappa_s(x)-8]}{(x-s)^3}dx\right| \\ & = \max_{x\in[x_{m-1},x_m]}|\mathcal{H}_m(x)|\int_{x_{m-1}}^{x_m}\frac{\kappa_s(x)-8}{(x-s)^3}\,dx \\ & = \max_{x\in[x_{m-1},x_m]}|\mathcal{H}_m(x)|\left\{\neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m}\frac{\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}\,dx - \neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m}\frac{8}{(x-s)^3}\,dx\right\} \\ & = \max_{x\in[x_{m-1},x_m]}|\mathcal{H}_m(x)|\left\{\frac{1}{\sin^2\frac{s-x_{m-1}}{2}}-\frac{1}{\sin^2\frac{s-x_m}{2}}+\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right\} \\ & = \max_{x\in[x_{m-1},x_m]}|\mathcal{H}_m(x)-\mathcal{H}_m(x_m)|\left\{\frac{1}{\sin^2\frac{s-x_{m-1}}{2}}-\frac{1}{\sin^2\frac{s-x_m}{2}}+\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right\} \\ & = \max_{x\in[x_{m-1},x_m]}|\mathcal{H}_m'(\xi_m)(s-x_m)|\left\{\frac{1}{\sin^2\frac{s-x_{m-1}}{2}}-\frac{1}{\sin^2\frac{s-x_m}{2}}+\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right\} \\ & \le C\gamma^{-2}(\tau)h^{l-2}. \end{aligned} \quad (40)$$

(35) can be obtained by putting together from (39) to (40) which completes the proof.

Lemma 3.8 Under the assumption of Theorem 2.1, we have

$$\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{x_{j-1}}^{x_j}\frac{\tilde{M}_l^j(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right|\leq C\max_{x\in(x_{j-1},x_j)}\{K_s(x)\}\gamma^{-2}(\tau)\frac{h^{l-2}}{l!} \tag{41}$$

and

$$\begin{aligned}&\left|\sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^j(x,s)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right|\\ &\leq\begin{cases}C\max\limits_{x\in(x_{j-1},x_j)}\{K_s(x)\}\gamma^{-1}(\tau)\dfrac{h^{l-2}}{(l-1)!}, & i=l-2,\\ C\max\limits_{x\in(x_{j-1},x_j)}\{K_s(x)\}\dfrac{h^{l-2}}{(l-2)!}(|\ln\gamma(\tau)|+|\ln h|), & i=l-3,\\ C\dfrac{h^{l-2}}{(l-i-1)!}, & i<l-3.\end{cases}\end{aligned} \tag{42}$$

Proof: By (33), we see that $|\tilde{\mathcal{M}}_l^j(x)|\leq Ch^{l+1}$, and thus

$$\begin{aligned}&\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{x_{j-1}}^{x_j}\frac{\tilde{M}_l^j(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right|\\ &=\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{x_{j-1}}^{x_j}\frac{\tilde{\mathcal{M}}_l^j(x)K_s(x)}{(x-s)^3}dx\right|\\ &\leq C\max_{x\in(x_{j-1},x_j)}\{K_s(x)\}\frac{h^l}{l!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\frac{1}{(x-s)^3}dx\\ &\leq C\max_{x\in(x_{j-1},x_j)}\{K_s(x)\}\gamma^{-2}(\tau)\frac{h^{l-2}}{l!}.\end{aligned} \tag{43}$$

Now, we estimate (42) and get

$$\begin{aligned}&\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^j(x,s)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right|\\ &=\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^{l-1}}{h(l-1)!}\int_{x_{j-1}}^{x_j}(f^{(l)}(\xi_{il})-f^{(l)}(s))\frac{\mathcal{M}_{l-2,l-1}^j(x,s)K_s(x)}{(x-s)^3}dx\right|\\ &\leq C\max_{x\in(x_{j-1},x_j)}\{K_s(x)\}\frac{h^{l-1}}{(l-1)!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\left|\frac{1}{(x-s)^2}\right|dt\\ &\leq C\max_{x\in(x_{j-1},x_j)}\{K_s(x)\}\gamma^{-1}(\tau)\frac{h^{l-2}}{(l-1)!}.\end{aligned} \tag{44}$$

and

$$\begin{aligned}&\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^j(x,s)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right|\\ &=\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^{l-1}}{h(l-1)!}\int_{x_{j-1}}^{x_j}(f^{(l)}(\xi_{il})-f^{(l)}(s))\frac{\mathcal{M}_{l-2,l-1}^j(x,s)K_s(x)}{(x-s)^3}dx\right|\\ &\leq C\max_{x\in(x_{j-1},x_j)}\{K_s(x)\}\frac{h^{l-1}}{(l-1)!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\left|\frac{1}{x-s}\right|dt\\ &\leq C\max_{x\in(x_{j-1},x_j)}\{K_s(x)\}(|\ln\gamma(\tau)|+|\ln h|)\frac{h^{l-2}}{(l-1)!}.\end{aligned} \tag{45}$$

The proof is completed.

Lemma 3.9Under the same assumption of Theorem 1 with $k=i,i+1,i+2$, there holds that

$$T_{ik}(\tau) = 8\sum_{j=-\infty}^{\infty} \psi_{ik}(2j+\tau). \tag{46}$$

Proof: By (23), we have

$$\begin{aligned} \int_{-1}^{1} \frac{\mathcal{M}_{ik}(\tau,t)\cos\frac{\tau-t}{2}}{\sin^3\frac{\tau-t}{2}}d\tau = \ & 8 \int_{-1}^{1} \frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t)^3}d\tau + 8\sum_{l=1}^{\infty}\int_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t-2l\pi)^3}d\tau \\ & +8\sum_{l=1}^{\infty}\int_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t+2l\pi)^3}d\tau, \end{aligned} \tag{47}$$

which means

$$\begin{aligned} T_{ik}(\tau) &= \sum_{i=1}^{n}\psi_{ik}(t) \\ &= 8\sum_{i=1}^{n}\phi_{ik}(2(m-i)+\tau) + 8\sum_{i=1}^{n}\sum_{l=1}^{\infty}\phi_{ik}(2(m-i-nl)+\tau) \\ &\quad + 8\sum_{i=1}^{n}\sum_{l=1}^{\infty}\phi_{ik}(2(m-i+nl)+\tau) \\ &= 8\sum_{l=-\infty}^{\infty}\sum_{i=1}^{n}\phi_{ik}(2(m-i+nl)+\tau) \\ &= 8[\phi_{ik}(\tau) + \sum_{l=1}^{\infty}\phi_{ik}(2l+\tau) + \sum_{l=1}^{\infty}\phi_{ik}(-2l+\tau)]. \end{aligned} \tag{48}$$

where

$$\phi_{ik}(t) = \begin{cases} \int_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t)^3}d\tau, & |t|<1, \\ \int_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t)^3}d\tau, & |t|>1, \end{cases} \tag{49}$$

which is related with second kind of legendre function. Since

$$Q_0(t) = \frac{1}{2}\log\left|\frac{1+t}{1-t}\right|, \tag{50}$$

then we have

$$\sum_{j=0}^{\infty}Q_0(2j+\tau) + \sum_{j=1}^{\infty}Q_0(-2j+\tau) = \frac{1}{2}\log\frac{2(n-m)-1+\tau}{2m+1-\tau} = 0. \tag{51}$$

By Lemma 3.1 and Lemma 3.2 in [11], we can easily show that $T_{ik}(\tau)$ converges to certain function.

Proof of Theorem 1:By Lemma 3.6, we have

$$\begin{aligned} & \left(\int_{c}^{x_{m-1}} + \int_{x_m}^{c+2\pi}\right)\frac{\cos\frac{x-s}{2}[f(x)-f_L(x)]}{\sin^3\frac{x-s}{2}}dx \\ = \ & \sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\frac{\cos\frac{x-s}{2}[f(x)-f_L(x)]}{\sin^3\frac{x-s}{2}}dx \\ = \ & \sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\frac{\cos\frac{x-s}{2}M_{ik}^{j}(x,s)}{\sin^3\frac{x-s}{2}}dx \\ & + \sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{\cos\frac{x-s}{2}M_{i,l-1}^{j}(x,s)}{\sin^3\frac{x-s}{2}}dx \\ & + \sum_{j=1,j\neq m}^{n}\frac{(-1)^n}{hl!}\int_{x_{j-1}}^{x_j}\frac{\cos\frac{x-s}{2}\tilde{M}_l^{j}(x)}{\sin^3\frac{x-s}{2}}dx. \end{aligned} \tag{52}$$

By the definition of $\mathcal{H}_m(x)$ in (34), we have

$$\begin{aligned}&\fint_{x_{m-1}}^{x_m} \frac{\cos\frac{x-s}{2}[f(x)-f_L(x)]}{\sin^3\frac{x-s}{2}}\,dx\\ &=\fint_{x_{m-1}}^{x_m} \frac{\cos\frac{x-s}{2}\mathcal{H}_m(x)}{\sin^3\frac{x-s}{2}}\,dx\\ &+\sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\fint_{x_{m-1}}^{x_m} \frac{\cos\frac{x-s}{2}M_{ik}^m(x,s)}{\sin^3\frac{x-s}{2}}\,dx \end{aligned}\tag{53}$$

Putting (52) and (53) together yields

$$\fint_c^{c+2\pi} \frac{\cos\frac{x-s}{2}[f(x)-f_L(x)]}{\sin^3\frac{x-s}{2}}dx = \sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^i h^{k-1}f^{(k+1)}(s)}{(i+1)!2^{k-1}}\sum_{j=1}^{n}\phi_{ik}(c_j)+R_f(s),$$

where

$$R_f(s)=\mathcal{R}^1(s)+\mathcal{R}^2(s)$$

$$\mathcal{R}^1(s)=\fint_{x_{m-1}}^{x_m}\frac{\mathcal{H}_m(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx,$$

$$\begin{aligned}\mathcal{R}^2(s)=&\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\tilde{M}_l^j(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}\,dx\\ &+\sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{x_{j-1}}^{x_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^j(t,s)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}\,dx.\end{aligned}$$

By Lemma 3.7 and Lemma 3.8, we have

$$|R_n(s)|\le C\max_{x\in(c,c+2\pi)}\{K_s(x)\}(|\ln h|+\gamma^{-2}(\tau))h^{l-2}.$$

The proof is complete.

Based on the Theorem 1, assume $f(x)\in C^4[c,c+2\pi]$, we present the modify trapezoidal rule

$$\tilde{I}_n(f;s)=I_n(f;s)-4\pi f''(s)\tan\frac{\tau\pi}{2},$$

and

$$\tilde{E}_n(f;s)=\fint_c^{c+2\pi}\frac{f(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx-\tilde{I}_n^2(f;s)$$

then we have

$$\tilde{E}_n(f;s)\le C\max_{x\in(c,c+2\pi)}\{K_s(x)\}[|\ln h|+\gamma^{-2}(\tau)]h^2$$

where $\gamma(\tau)$ is defined as (19).

4.Extrapolation method

In the above sections, we have proved that the error functional of the trapezoidal rule has the following asymptotic expansion of (17). We present our extrapolation algorithms as follow:
Assume there exists positive integer n_0 such that

$$m_0:=\frac{n_0(s-c)}{2\pi}$$

is a positive number. Firstly $[c,c+2\pi]$ is partitioned into n_0 equal subinterval denoted by Π_1 with mesh size $h_1=2\pi/n_0$. Then we refine Π_1 to get mesh Π_2 with mesh size $h_2=h_1/2$. In this way, a series of meshes $\{\Pi_j\}(j=1,2,\cdots)$ in which Π_j is refined from Π_{j-1} with mesh size denoted by Π_{j-1}. Define

$$s_j = s + \frac{h_j}{2}, j = 1, 2, \cdots . \tag{54}$$

and

$$T(h_j) = I_{2^{j-1}n_0}(f, s_j).$$

We present the following extrapolation algorithm: first compute

$$T_1^{(j)} = T(h_j), j = 1, \cdots, m.$$

Second compute

$$T_i^{(j)} = T_{i-1}^{(j+1)} + \frac{T_{i-1}^{(j+1)} - T_{i-1}^{(j)}}{2^{i-1} - 1}, \quad i = 2, \cdots, m \quad j = 1, \cdots, m - i.$$

Thoerem 2 Under the asymptotic expansion of theorem 2.1, for $\tau = 0$ and the series of meshes defined by (54), we have

$$\left| I(f, s) - T_i^{(j)} \right| \le Ch^i$$

and a posteriori asymptotic error estimate is given by

$$\left| \frac{T_{i-1}^{(j+1)} - T_{i-1}^{(j)}}{2^{i-1} - 1} \right| \le Ch^{i-1}.$$

5.Numerical example

In this section, computational results are reported to confirm our theoretical analysis.

Example 1 Consider the supersingular integral

$$\int_c^{c+2\pi} \frac{\cos\frac{x-s}{2} f(x)}{\sin^3\frac{x-s}{2}} dx = g(s), s \in (c, c+2\pi)$$

with $f(x) = 1 + \sin(3x) + \cos(2x)$ and the exact analysis is $4\pi(-9cos(3s) + 4sin(2s))$

Table1 Errors of the trapezoidal rule rules $s = x_{[n/4]} + (1+\tau)h/2$

n	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = 1/2$
32	8.1300e-01	8.7751e+01	-1.0333e+02	5.3026e+01
64	1.0380e-01	1.0320e+02	-1.0747e+02	6.0147e+01
128	1.3044e-02	1.0739e+02	-1.0849e+02	6.2136e+01
256	1.6327e-03	1.0846e+02	-1.0874e+02	6.2655e+01
512	2.0416e-04	1.0874e+02	-1.0881e+02	6.2787e+01
1024	2.5555e-05	1.0881e+02	-1.0882e+02	6.2821e+01
h^α	2.9915	-	-	-

Table2 Errors of the mod-trapezoidal rule $s = x_{[n/4]} + (1+\tau)h/2$

n	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = 1/2$
32	8.1300e-01	-2.5666e+00	4.7362e+00	-1.1117e+00
64	1.0380e-01	-9.2221e-01	1.1658e+00	-4.8090e-01
128	1.3044e-02	-2.5972e-01	2.8810e-01	-1.4320e-01
256	1.6327e-03	-6.8134e-02	7.1544e-02	-3.8476e-02
512	2.0416e-04	-1.7405e-02	1.7822e-02	-9.9397e-03
1024	2.5555e-05	-4.3961e-03	4.4477e-03	-2.5241e-03
h^α	2.9915	1.8379	2.0113	1.7566

Table3 Errors of the trapezoidal rule $s = x_0 + (1+\tau)h/2$

n	$\tau=0$	$\tau=2/3$	$\tau=-2/3$	$\tau=1/2$
32	3.2188e+00	5.2573e+00	-6.3854e+01	3.6217e+00
64	8.2217e-01	3.9187e+01	-7.6374e+01	2.5780e+01
128	2.0567e-01	6.2972e+01	-8.1977e+01	3.7852e+01
256	5.1301e-02	7.5017e+01	-8.4588e+01	4.4033e+01
512	1.2802e-02	8.1044e+01	-8.5843e+01	4.7145e+01
1024	3.1973e-03	8.4055e+01	-8.6457e+01	4.8705e+01
h^α	1.9951	-	-	-

Table4 Errors of the mod-trapezoidal rule $s=x_0+(1+\tau)h/2$

n	$\tau=0$	$\tau=2/3$	$\tau=-2/3$	$\tau=1/2$
32	3.2188e+00	4.6428e+00	3.8210e+00	3.8759e+00
64	8.2217e-01	8.8449e-01	1.0294e+00	8.3812e-01
128	2.0567e-01	1.7963e-01	2.6605e-01	1.8823e-01
256	5.1301e-02	3.9364e-02	6.7560e-02	4.4115e-02
512	1.2802e-02	9.1278e-03	1.7019e-02	1.0645e-02
1024	3.1973e-03	2.1907e-03	4.2717e-03	2.6124e-03
h^α	1.9951	2.2099	1.9610	2.1070

For the case of $s=x_{[n/4]}+(1+\tau)h/2$,Table1 show that when the local coordinate of singular point $\tau=0$, the quadrature reach the convergence rate of $O(h^2)$ as for the non-supersingular point the the convergence rate which agree with our theorematically analysis. From the Table 2 shows the modify trapezoidal rule have the convergence rate of $O(h^2)$ at both the superconvergence point and non-superconvergence point which coincide with our results.For the case of $s=x_0+(1+\tau)h/2$because of no influence of the boundary condition, from table 3 and table 4, we get the superconvergence point the same as $s=x_{[n/4]}+(1+\tau)h/2$ and the superconvergence rate as following which coincide with our theoretically analysis.

Example 2 We still consider the supersingular integral as example 1
with $f(x)=1+\sin(3x)+\cos(2x)$ and the exact analysis is $4\pi(-9cos(3s)+4sin(2s))$

Table 5 Errors of the linear trapezoidal rule $s=-\pi/2$

	$\tau=0$	h^2-extrapolation	h^3-extrapolation	h^4-extrapolation
32	8.1836e+00			
64	3.7710e+00	-6.4164e-01		
128	1.7978e+00	-1.7536e-01	-1.9940e-02	
256	8.7613e-01	-4.5544e-02	-2.2699e-03	2.5439e-04
512	4.3227e-01	-1.1587e-02	-2.6881e-04	1.7053e-05
1024	2.1467e-01	-2.9214e-03	-3.2661e-05	1.0752e-06

Table 6 Errors of the linear trapezoidal rule $s=-\pi/2$

	$\tau=0$	h^2-extrapolation	h^3-extrapolation	h^4-extrapolation
32				
64	4.4126e+00			
128	1.9732e+00	-1.5542e-01		
256	9.2167e-01	-4.3274e-02	-2.5243e-03	
512	4.4386e-01	-1.1319e-02	-2.8587e-04	1.5822e-05
1024	2.1760e-01	-2.8887e-03	-3.3736e-05	1.0652e-06

Now we consider $s=-\pi/2$ and choose the approximation series $s_i=-\frac{\pi}{2}+\frac{\tau+1}{2}h_i$with starting meshes $n_0=32$, then s can be located at the mesh point. We list the error and a posteriori estimate in table 5 and table 6 with the extrapolation rate is h, h^2, h^3 and h^4, respectively. From the table we can see that the numerical results agree with the theoretical analysis very well.

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