High Order Discontinuous Galerkin Method for the Euler Equations Using Curved Elements

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Abstract

In this study, a high order discontinuous Galerkin method for the two dimensional Euler equations is presented, the physical domain is divided into triangular elements which form an unstructured mesh. High order 6-node triangular elements with curved edges are introduced for curved physical boundaries. Polynomial functions up to fourth order are used as basis functions in each computational element. Fluxes between elements are calculated using HLLC approximate Riemann solver. An explicit third order Runge-Kutta time integration method is employed to solve the discretized systems. A number of test cases are presented to demonstrate the accuracy of this method. The results show that when curved elements are utilized, this method could archive its designed accuracy on domains with curved geometry boundaries.

Keywords: Discontinuous Galerkin Method, Euler equations, unstructured mesh.

Introduction

The discontinuous Galerkin method (DGM) was first proposed by Reed and Hill[1] for solving neutron transportation problems, since then, the DGM are extensively used in many areas, which include fluid simulations, MHD simulations, shallow water simulations and many others. In the field of computational fluid dynamics, finite difference method (FDM) and finite volume method (FVM) have long history of applications. FDM is suitable for building high order numerical schemes, but it has many difficulties when dealing with complex geometries and unstructured meshes. FVM could be implemented on unstructured meshes and complex geometries easily, but it is hard to construct high order compact FVM schemes. The DGM has both the advantages of FDM and FVM. The discrete unknown variables of DGM are linear combinations of element-wise polynomial basis functions, high order DGM schemes could be built on arbitrary meshes with compact stencils.

High order DGM is much more sensitive to the numerical boundary conditions than other methods[2,3]. When the physical boundaries are curved, boundary elements with straight edges may not meet the demand of DGM and lead to unphysical solutions. In these cases, accurate representations of curved boundaries are crucial for performing high order DGM simulations.

The present authors have developed a 2D DGM for Euler equations on unstructured meshes. High order elements with curved edges are utilized at physical boundaries. Numerical tests show that when curved elements are utilized, DGM could archive its designed accuracy on domains with curved physical boundaries.

Governing Equations

The conservative forms of Euler equations are

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0 \tag{1}$$

Where *U*, *F* and *G* refer to conservative state vector, *x*-direction inviscid flux and *y*-direction inviscid flux respectively.

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{bmatrix};\ F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u\left(\rho E+p\right)\end{bmatrix};\ G=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v\left(\rho E+p\right)\end{bmatrix} \tag{2}$$

To enclose the equation system, the equation of state is introduced.

$$p=\rho RT \tag{3}$$

Discontinuous Galerkin Method

The physical domain is divided into non-overlapping elements *K*. The unknowns U_h on each element are expressed as linear combinations of basis functions.

$$U_h=U_i\phi_i \tag{4}$$

In each element, the weak form of governing equations is introduced.

$$\int_{K_j}\left(\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}\right)\phi_i \mathrm{d}\Omega=0 \tag{5}$$

With some manipulations, the equations have the following form.

$$\int_{K_j}\frac{\partial U}{\partial t}\phi_i \mathrm{d}\Omega-\int_{K_j}\left(F\frac{\partial \phi_i}{\partial x}+G\frac{\partial \phi_i}{\partial y}\right)\mathrm{d}\Omega+\int_{\partial K_j}\phi_i\ \ Fn_x+Gn_y\ \ \mathrm{d}S=0 \tag{6}$$

The solution of DGM has multiple values on element boundaries. In order to determine a unique value of inter-cell fluxes, numerical flux functions are introduced, in this study, the HLLC Riemann solver[4] is adopted.

At each time step, a system of ordinary differential equation is formed

$$M_{K_j}\frac{dU}{dt}=R \tag{7}$$

Where *R* is residual term, *M* refers to the mass matrix on element K_j. A third order explicit Runge-Kutta scheme is utilized to solve this ordinary equation system.

Numerical Results

The 2D subsonic flow around a cylinder[5] at Mach number 0.38 is chosen to demonstrate the performance of DGM with the existence of curved boundaries. The radius of cylinder is 0.5, computational domain is bounded by a circle of r=20. four successively refined meshes are generated, which contains 16×4, 32×8, 64×16 and 128×32 points. Details about these meshes could be found in [6], the meshes are shown in Fig.1.

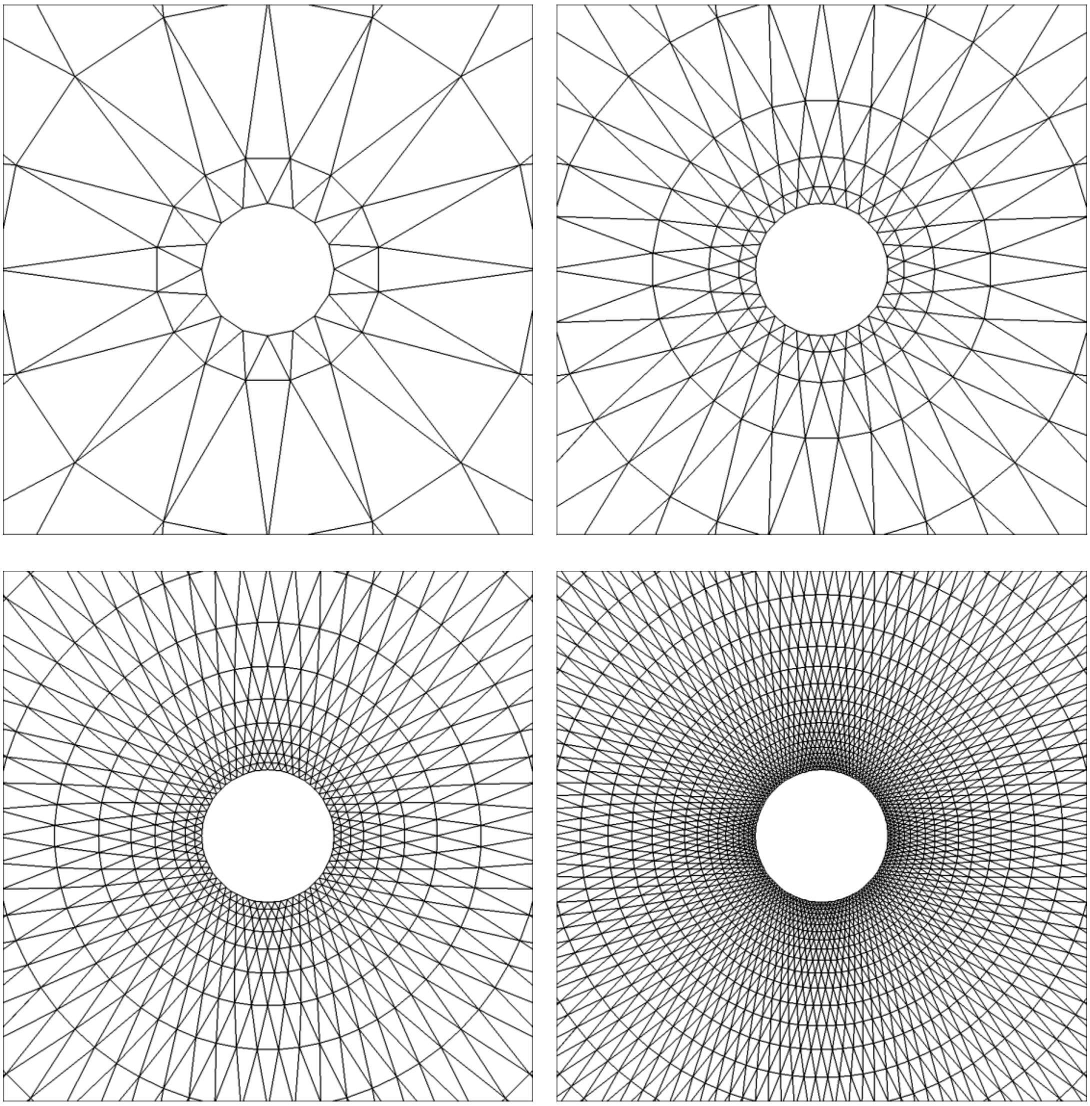
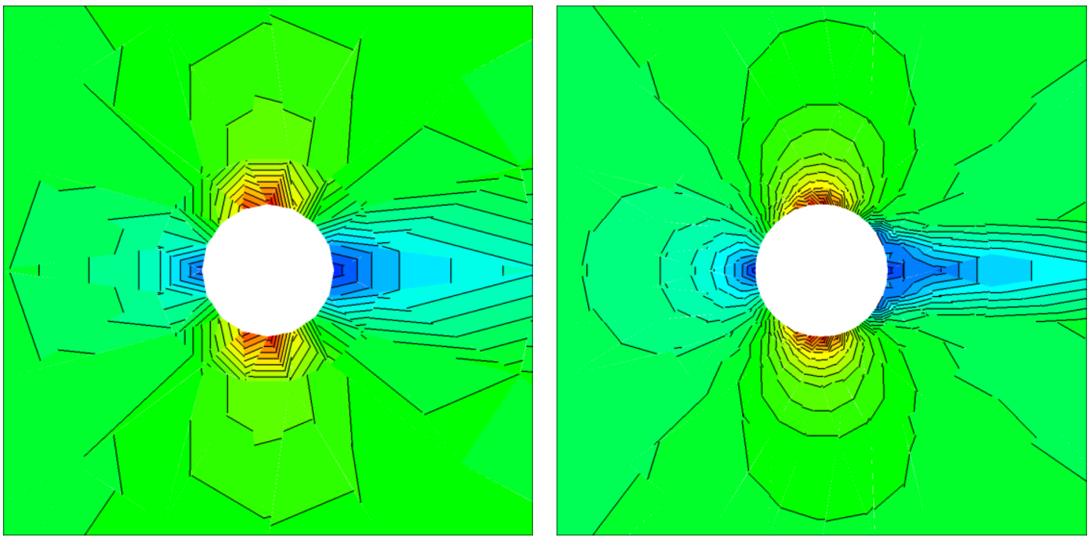


Figure 1. Successively refined meshes of cylinder, 16×4 (top left), 32×8 (top right), 64×16 (bottom left) and 128×32 (bottom right)



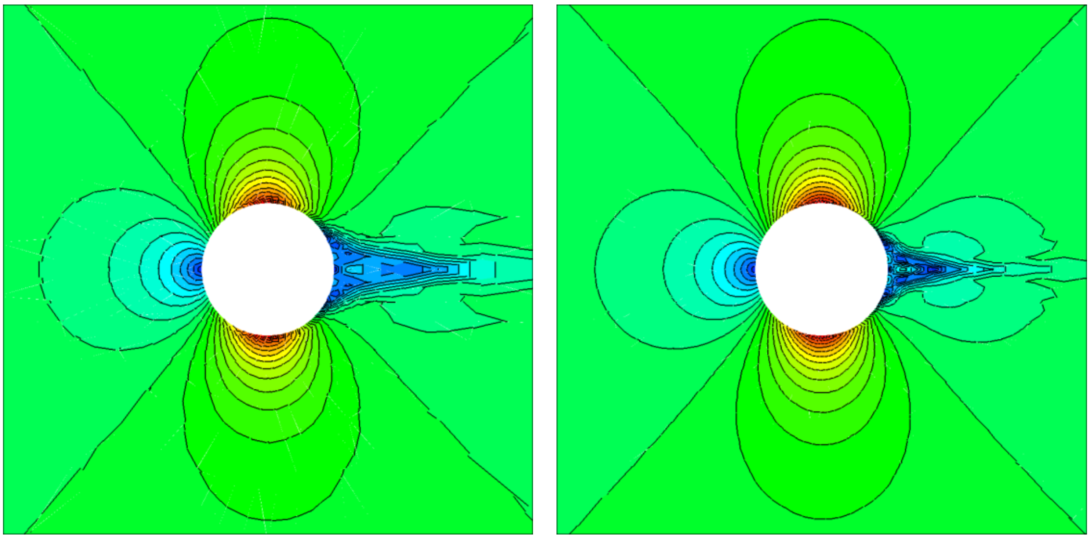


Figure 2. Mach contour using mesh 16×4 (top left), 32×8 (top right), 64×16 (bottom left) and 128×32 (bottom right), with straight-edge triangular elements

Mach contours using these meshes of first order DGM are shown in Fig.2, ΔM=0.038. Theatrically, the contour should have a symmetric pattern, but all of these results show unphysical wakes along the downstream of cylinder, and these unphysical wakes will not disappear with the successive refinement of meshes. This indicates that straight edged elements could not fit curved boundaries and cause wrong solutions.

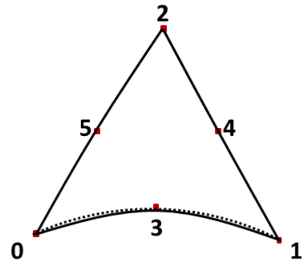


Figure 3. 6-node triangular element (solid) fitting a curved boundary (dashed)

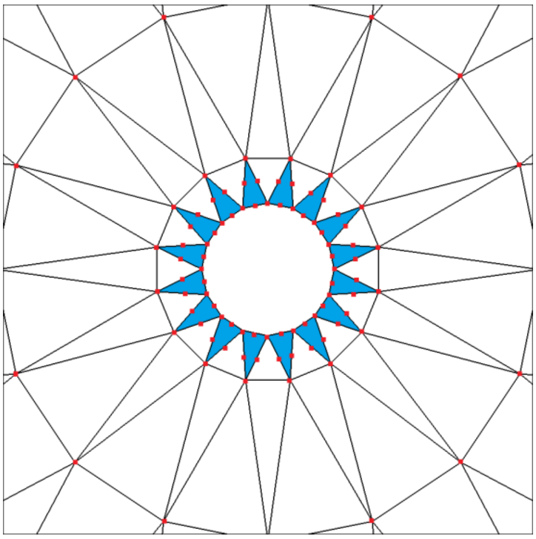


Figure 4. Mesh with 6-node curved triangular elements (blue) near boundary

In order to generate meshes for curved boundaries, the high order 6-node curve edged triangular element is introduced, and it is shown in Fig.3. With curved elements at cylinder boundary, a new mesh is generated (Fig.4). This mesh contains both curve edged elements for boundary fitting and straight edged elements for space filling.

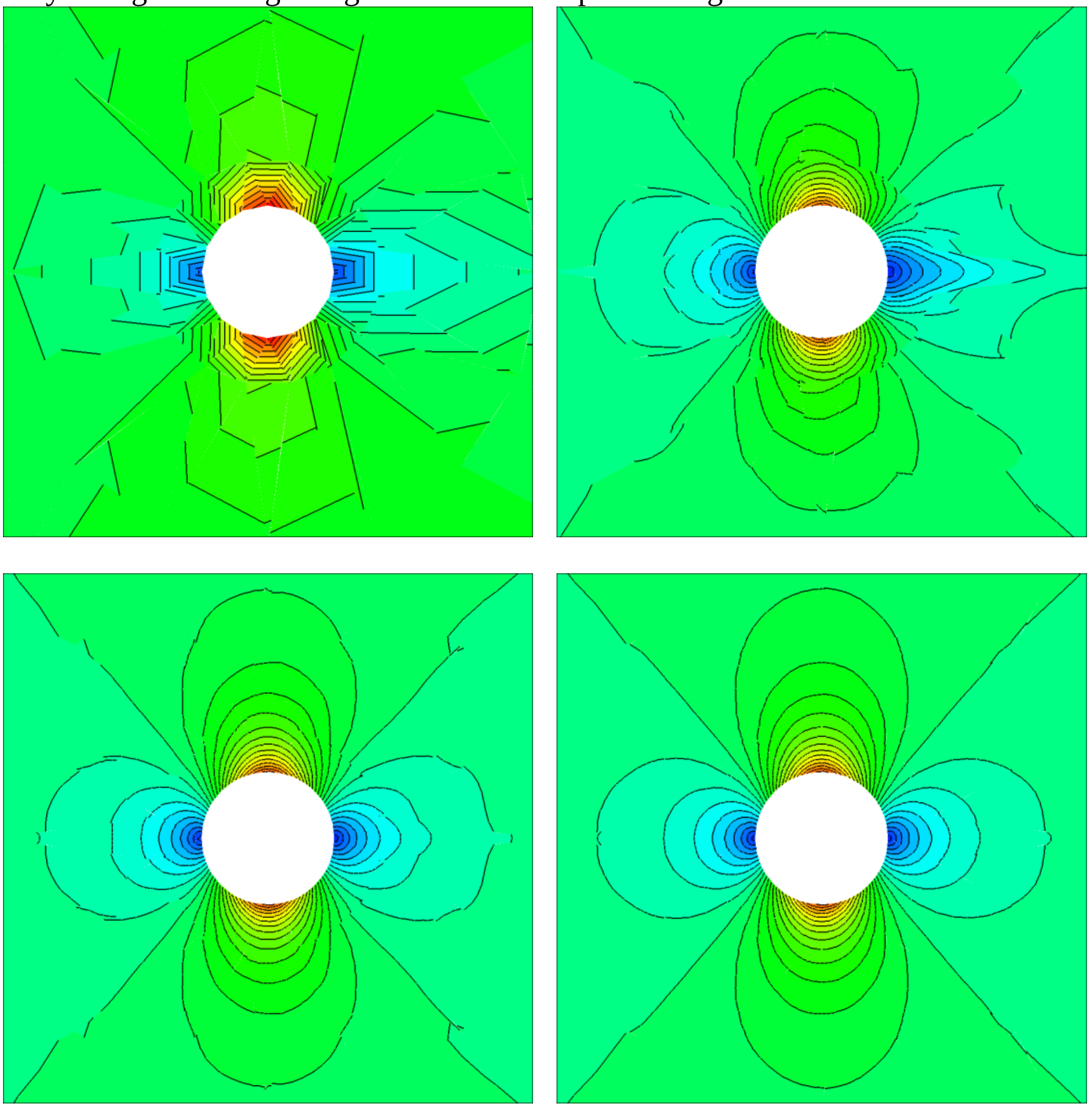


Figure 5. Mach contour on 16×4 mesh with p=1 (top left) p=2 (top right), p=3 (bottom left) and p=4 (bottom right), with curved boundary elements

Mach contours obtained with up to fourth order DGM using this new mesh are shown in Fig.5. The results show that, with the utilizations of curved elements near cylinder boundary, numerical dissipations are reduced, and the symmetric pattern of Mach contour is obtained with high order DGM, which implies the correct discretization of curved boundaries with high order curved elements is crucial in high order DGM computations.

Conclusions

In this study, a discontinuous Galerkin method with curved boundary treatment is presented. The curved solid boundary is fitted with high order 6-node triangular elements. The results of subsonic cylinder flows show that, DGM is very sensitive to the treatment of curved boundaries, high order curved elements must be utilized to archive its designed accuracy on domains with curved physical boundaries.

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