Static and free vibration of laminated composite plates using higher order cell-based smoothed finite element method with Q8 elements

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Abstract

In this paper, the higher order cell-based smoothed finite element method based on the first-order shear deformation theory is used for the analysis of laminated composite plates. The domain is discretized with eight-node Mindlin plate elements of serendipity family (Q8 elements). Higher order finite element with Q8 elements using the selectively reduced integration is known to alleviate the shear-locking phenomenon. However, it still produces shear-locking phenomenon below a certain thickness-span ratio and also yields poor solutions and sub-optimal convergence rates with distorted meshes. In this paper, we propose a novel approach to eradicate the shear-locking phenomenon and improve the quality of the solutions by employed a linear smoothing technique. Within this technique, each Q8 element is subdivided into eight smoothing cells, and a modified strain is computed over the smoothing cell by using a linear smoothing procedure. The modified bending strain and shear strain are computed by the divergence theorem between the nodal shape functions and their derivatives in Taylor's expansion within each smoothing cell. Several numerical examples indicate that the novel approach can eradicate the shear-locking and also yield more reliable results for the distorted meshes.

Keywords: Laminated composite plates, S-FEM, linear smoothing technique, shear-locking, distorted meshes.

Introduction

Due to the remarkable weight-to-stiffness, strength-to-stiffness characteristics and the advantage of designable, laminated composite plates are widely used in engineering structures as diverse as aerospace, aircrafts, automotive structural parts, civil engineering structures, etc. Many plate theories have been successfully applied to analyze the laminated composite plates. The first-order shear deformation theory is widely used in the analysis of composite plates [1]. In addition, Hinton and Zienkiewicz [2, 3] explicitly indicated that the eight-node Mindlin plate elements of serendipity family (Q8 elements) still suffer from the shear-locking phenomenon below a certain thickness-span ratio and yields poor solutions of distorted meshes.

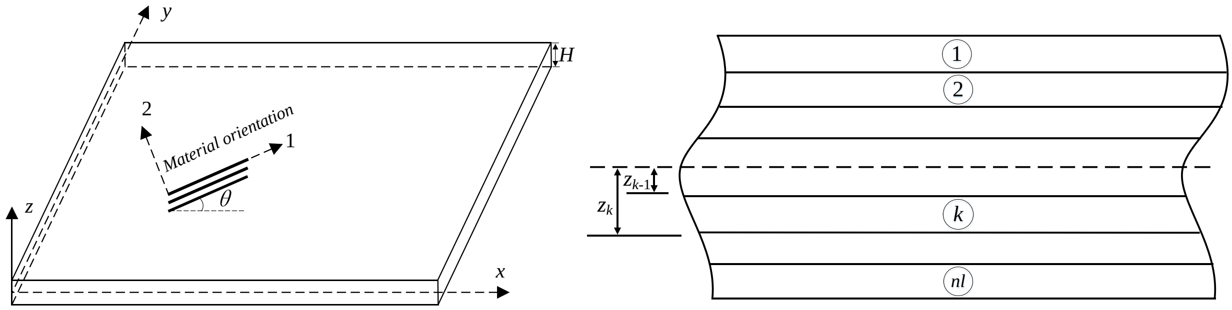
In order to eliminate shear-locking phenomenon and improve the quality of finite element solutions over simplex elements, recently, Liu et al. [4]-[6] proposed a smoothed finite element method (S-FEM), which is based on the stabilized conforming nodal integration (SCNI) of mesh-free method [7], Note that all the types of S-FEM use finite element with linear interpolants, and the strain smoothing technique over the higher order elements exhibits poor performance [8]. Francis et al. [9] proposed a linear strain smoothing scheme with the framework of CS-FEM to improve accuracy of arbitrary convex polytopes with linear or quadratic interpolants, which was based on the recent work of Duan et al. [10].

In this paper, with the aim of eliminating the shear-locking phenomenon and improving the accuracy of the solution with distorted meshes, a higher order CS-FEM with eight-node Mindlin plate elements is developed based on the first-shear deformation theory, which is a further application of the linear smoothing technique. With this technique, as shown in Fig .1, each Q8 element is subdivided into eight smoothing cells, and the modified bending strain and shear strain are computed by the divergence theorem between the nodal shape functions and their derivatives in Taylor's expansion within each smoothing cell. This eliminates the need for isoparametric mapping and all the domain integration are transformed into boundary integration. Meanwhile, the proposed approach can effectively treat the shear-locking phenomenon for both thin and relatively thick plates, and the effect of the mesh distortion to the accuracy can be relieved.

Basic formulations

According to the first-shear deformation theory, the displacements field of the laminated composite plates (shown in **Fig. 1**) can be expressed as

$$\begin{cases} u(x,y,z)=u_0(x,y)-z\beta_x(x,y) \\ v(x,y,z)=v_0(x,y)-z\beta_y(x,y) \\ w(x,y,z)=w_0(x,y) \end{cases} \tag{1}$$



(a) (b)

Fig. 1. (a) Laminated composite plate axes system. (b) Layer details.

The modified strain is given by

$$\tilde{\boldsymbol{\varepsilon}}(\mathbf{x}_k)=\int_{\Omega_s}\boldsymbol{\varepsilon}(\mathbf{x})\mathbf{q}(\mathbf{x})\mathrm{d}\Omega \tag{2}$$

In the process of computing the modified strain matrix $\mathbf{B}_I^b$ and $\mathbf{B}_I^s$, the consistency form should be met by the terms related the shape function $N_I(\mathbf{x})$ and the derivative of shape function $N_{I,i}(\mathbf{x})$ $(i=x,y)$. According to Eq. (2) and implementing the divergence theorem, we can obtain

$$\int_{\Omega_s}N_{I,i}\mathbf{q}(\mathbf{x})\mathrm{d}\Omega=\int_{\Gamma_s}N_I\mathbf{q}(\mathbf{x})n_i\mathrm{d}\Gamma-\int_{\Omega_s}N_I\mathbf{q}_{,i}(\mathbf{x})\mathrm{d}\Omega \tag{3}$$

$$\mathbf{q}(\mathbf{x})=\mathbf{q}(\mathbf{x}_c)+(x-x_c)\mathbf{q}_{,x}(\mathbf{x}_c)+(y-y_c)\mathbf{q}_{,y}(\mathbf{x}_c)+\mathrm{H.O.T} \tag{4}$$

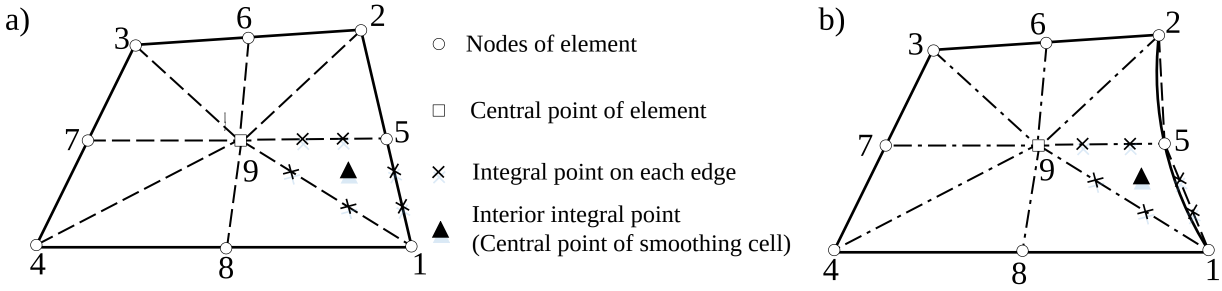


Fig. 2. Division of a Q8 plate element into eight smoothing cells

The evaluation of the modified derivatives $\tilde{N}_{I,x}(\mathbf{x})$ is firstly presented. To begin with, $\tilde{N}_{I,x}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ are treated by Taylor's expansion, and the expanded form of $\tilde{N}_{I,x}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ are expressed as

$$\tilde{N}_{I,x}(\mathbf{x}) = \tilde{N}_{I,x}(\mathbf{x}_c) + (x - x_c)\tilde{N}_{I,xx}(\mathbf{x}_c) + (y - y_c)\tilde{N}_{I,xy}(\mathbf{x}_c) + \text{H.O.T} \tag{5}$$

$$\mathbf{q}(\mathbf{x}) = \mathbf{q}(\mathbf{x}_c) + (x - x_c)\mathbf{q}_{,x}(\mathbf{x}_c) + (y - y_c)\mathbf{q}_{,y}(\mathbf{x}_c) + \text{H.O.T} \tag{6}$$

where H.O.T is the higher order term. $N_{I,x}(\mathbf{x})$ is replaced by $\tilde{N}_{I,x}(\mathbf{x})$, then, substituting Eqs. (5) and (6) into Eq. (3), whereafter, one interior integral point is used for the left hand side of Eq. (3), and two integral point on each edge of the smoothing cell are employed for the right hand side of Eq. (3), and the subdivision of a Q8 element into smoothing cells is shown in **Fig. 2.** We can obtain

$$\begin{aligned}&\left[\mathbf{q}(\mathbf{x}_c)A\right]\tilde{N}_{I,x}(\mathbf{x}_c) + \left[\mathbf{q}_{,x}(\mathbf{x}_c)I_c^{xx} + \mathbf{q}_{,y}(\mathbf{x}_c)I_c^{xy}\right]\tilde{N}_{I,xx}(\mathbf{x}_c) + \left[\mathbf{q}_{,x}(\mathbf{x}_c)I_c^{xy} + \mathbf{q}_{,y}(\mathbf{x}_c)I_c^{yy}\right]\tilde{N}_{I,xy}(\mathbf{x}_c) \\ &= \int_{\Gamma_s} N_I(\mathbf{x})\mathbf{q}(\mathbf{x})n_x \mathrm{d}\Gamma - \int_{\Gamma_s}\left(\frac{\partial\left(\int N_I(\mathbf{x})\mathbf{q}_{,x}(\mathbf{x})\mathrm{d}x\right)}{\partial x}\right)n_x \mathrm{d}\Gamma\end{aligned} \tag{7}$$

where A is the area of the smoothing cell, and

$$\begin{cases} I_c^{xx} = \int_{\Omega_s}(x - x_c)^2 \mathrm{d}\Omega \\ I_c^{xy} = \int_{\Omega_s}(x - x_c)(y - y_c)\mathrm{d}\Omega \\ I_c^{xy} = \int_{\Omega_s}(y - y_c)^2 \mathrm{d}\Omega \end{cases} \tag{8}$$

Substituting Eq.(4) into Eq.(7) and the two Gauss points on each edge is used for the evaluation of the boundary integral, therefore, the expanded form is expressed as

$$\begin{bmatrix} A & 0 & 0 \\ Ax_c & I_c^{xx} & I_c^{xy} \\ Ay_c & I_c^{xy} & I_c^{yy} \end{bmatrix} \begin{Bmatrix} \tilde{N}_{I,x}(\mathbf{x}_c) \\ \tilde{N}_{I,xx}(\mathbf{x}_c) \\ \tilde{N}_{I,xy}(\mathbf{x}_c) \end{Bmatrix} = \begin{Bmatrix} F_{11}^x \\ F_{12}^x \\ F_{13}^x \end{Bmatrix} \tag{9}$$

where

$$\begin{Bmatrix} F_{11}^x \\ F_{12}^x \\ F_{13}^x \end{Bmatrix} = \begin{Bmatrix} \sum_{k=1}^{N_{eg}}\sum_{G=1}^{2} N_I(\mathbf{x}_G^k)n_x^k W_G \\ \sum_{k=1}^{N_{eg}}\sum_{G=1}^{2}\left(N_I(\mathbf{x}_G^k)x_G^k - \left(\frac{\partial\left(\int N_I(\mathbf{x}_G^k)\mathrm{d}x\right)}{\partial x}\right)\right)n_x^k W_G \\ \sum_{i=1}^{N_{eg}}\sum_{G=1}^{2} N_I(\mathbf{x}_G^k)y_G^k n_x^k W_G \end{Bmatrix} \tag{10}$$

Then, the modified derivatives can be obtained by analytically solving Eq. (9). Therefore, $\tilde{N}_{I,x}(\mathbf{x}_c)$, $\tilde{N}_{I,xx}(\mathbf{x}_c)$ and $\tilde{N}_{I,xy}(\mathbf{x}_c)$ are given as

$$\begin{cases} \tilde{N}_{I,x}(\mathbf{x}_c) = \dfrac{F_{11}^x}{A} \\ \tilde{N}_{I,xy}(\mathbf{x}_c) = \dfrac{I_c^{yy}\left(F_{12}^x - F_{11}^x x_c\right) - I_c^{xy}\left(F_{13}^x - F_{11}^x y_c\right)}{I_c^{xx} I_c^{yy} - \left(I_c^{xy}\right)^2} \\ \tilde{N}_{I,xx}(\mathbf{x}_c) = \dfrac{I_c^{xx}\left(F_{13}^x - F_{11}^x y_c\right) - I_c^{xy}\left(F_{12}^x - F_{11}^x x_c\right)}{I_c^{xx} I_c^{yy} - \left(I_c^{xy}\right)^2} \end{cases} \tag{11}$$

In such a way, the evaluation of the modified derivatives, $\tilde{N}_{I,y}(\mathbf{x}_c)$, $\tilde{N}_{I,yx}(\mathbf{x}_c)$ and $\tilde{N}_{I,yy}(\mathbf{x}_c)$ can be obtained.

After that, the modified bending and shear strain matrices, and the modified derivatives of the bending and shear strain matrices can be respectively expressed as

$$\tilde{\mathbf{B}}_I^b = \begin{bmatrix} \tilde{N}_{I,x} & 0 & 0 & 0 & 0 \\ 0 & \tilde{N}_{I,y} & 0 & 0 & 0 \\ \tilde{N}_{I,y} & \tilde{N}_{I,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{N}_{I,x} & 0 \\ 0 & 0 & 0 & 0 & \tilde{N}_{I,y} \\ 0 & 0 & 0 & \tilde{N}_{I,\mathrm{y}} & \tilde{N}_{I,x} \end{bmatrix}, \quad I = 1 \cdots 8 \tag{12}$$

$$\frac{\partial \tilde{\mathbf{B}}_I^b}{\partial x} = \begin{bmatrix} \tilde{N}_{I,xx} & 0 & 0 & 0 & 0 \\ 0 & \tilde{N}_{I,yx} & 0 & 0 & 0 \\ \tilde{N}_{I,yx} & \tilde{N}_{I,xx} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{N}_{I,xx} & 0 \\ 0 & 0 & 0 & 0 & \tilde{N}_{I,yx} \\ 0 & 0 & 0 & \tilde{N}_{I,\mathrm{yx}} & \tilde{N}_{I,xx} \end{bmatrix}, \quad I = 1 \cdots 8 \tag{13}$$

$$\frac{\partial \tilde{\mathbf{B}}_I^b}{\partial y} = \begin{bmatrix} \tilde{N}_{I,xy} & 0 & 0 & 0 & 0 \\ 0 & \tilde{N}_{I,yy} & 0 & 0 & 0 \\ \tilde{N}_{I,yy} & \tilde{N}_{I,xy} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{N}_{I,xy} & 0 \\ 0 & 0 & 0 & 0 & \tilde{N}_{I,yy} \\ 0 & 0 & 0 & \tilde{N}_{I,\mathrm{yy}} & \tilde{N}_{I,xy} \end{bmatrix}, \quad I = 1 \cdots 8 \tag{14}$$

$$\tilde{\mathbf{B}}_I^s = \begin{bmatrix} 0 & 0 & \tilde{N}_{I,x} & -\tilde{N}_I & 0 \\ 0 & 0 & \tilde{N}_{I,\mathrm{y}} & 0 & \tilde{N}_I \end{bmatrix}, \quad I = 1 \cdots 8 \tag{15}$$

$$\frac{\partial \tilde{\mathbf{B}}_I^s}{\partial x} == \begin{bmatrix} 0 & 0 & \tilde{N}_{I,xx} & -\tilde{N}_{I,x} & 0 \\ 0 & 0 & \tilde{N}_{I,\mathrm{yx}} & 0 & \tilde{N}_{I,x} \end{bmatrix}, \quad I = 1 \cdots 8 \tag{16}$$

$$\frac{\partial \tilde{\mathbf{B}}_I^s}{\partial y} == \begin{bmatrix} 0 & 0 & \tilde{N}_{I,xy} & -\tilde{N}_{I,y} & 0 \\ 0 & 0 & \tilde{N}_{I,\mathrm{yy}} & 0 & \tilde{N}_{I,\mathrm{y}} \end{bmatrix}, \quad I = 1 \cdots 8 \tag{17}$$

By using the Taylor's expansion with respect to the each smoothing cell center $\mathbf{x}_c$, smoothed stiffness matrix are rewritten as

$$
\begin{aligned}
\left(\tilde{\mathbf{K}}^{b}\right)^{h} &= \sum_{C=1}^{nc}\left(\int_{\Omega_s} \tilde{\mathbf{B}}^{b\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{B}}^{b}\mathrm{d}\Omega\right) \\
&= \sum_{C=1}^{nc}\left(\int_{\Omega_s}\left[\tilde{\mathbf{B}}^{b\mathrm{T}}+\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial x}\left(x-x_c\right)+\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial y}\left(y-y_c\right)\right]\bar{\mathbf{D}}\left[\tilde{\mathbf{B}}^{b}+\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial x}\left(x-x_c\right)+\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial y}\left(y-y_c\right)\right]\mathrm{d}\Omega\right) \\
&= \sum_{C=1}^{nc}\left(A\tilde{\mathbf{B}}^{b\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{B}}^{b\mathrm{T}}+I_c^{xx}\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial x}\bar{\mathbf{D}}\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial x}+I_c^{yy}\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial y}\bar{\mathbf{D}}\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial y}+I_c^{xy}\left(\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial x}\bar{\mathbf{D}}\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial y}+\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial y}\bar{\mathbf{D}}\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial x}\right)\right)
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\left(\tilde{\mathbf{K}}^{b}\right)^{h} &= \sum_{C=1}^{nc}\left(\int_{\Omega_s} \tilde{\mathbf{B}}^{b\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{B}}^{b}\mathrm{d}\Omega\right) \\
&= \sum_{C=1}^{nc}\left(\int_{\Omega_s}\left[\tilde{\mathbf{B}}^{b\mathrm{T}}+\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial x}\left(x-x_c\right)+\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial y}\left(y-y_c\right)\right]\bar{\mathbf{D}}\left[\tilde{\mathbf{B}}^{b}+\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial x}\left(x-x_c\right)+\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial y}\left(y-y_c\right)\right]\mathrm{d}\Omega\right) \\
&= \sum_{C=1}^{nc}\left(A\tilde{\mathbf{B}}^{b\mathrm{T}}\bar{\mathbf{D}}\tilde{\mathbf{B}}^{b\mathrm{T}}+I_c^{xx}\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial x}\bar{\mathbf{D}}\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial x}+I_c^{yy}\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial y}\bar{\mathbf{D}}\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial y}+I_c^{xy}\left(\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial x}\bar{\mathbf{D}}\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial y}+\frac{\partial\tilde{\mathbf{B}}^{b\mathrm{T}}}{\partial y}\bar{\mathbf{D}}\frac{\partial\tilde{\mathbf{B}}^{b}}{\partial x}\right)\right)
\end{aligned}
\tag{19}
$$

where

$$\tilde{\mathbf{B}}^{b}=[\tilde{\mathbf{B}}_1^{b} \quad \tilde{\mathbf{B}}_2^{b} \quad \cdots \quad \tilde{\mathbf{B}}_8^{b}] \tag{20}$$

and

$$\tilde{\mathbf{B}}^{s}=[\tilde{\mathbf{B}}_1^{s} \quad \tilde{\mathbf{B}}_2^{s} \quad \cdots \quad \tilde{\mathbf{B}}_8^{s}] \tag{21}$$

Finally, the smoothed stiffness matrix $\tilde{\mathbf{K}}_{IJ}$ are rewritten as

$$\tilde{\mathbf{K}}_{IJ}=\tilde{\mathbf{K}}_{IJ}^{b}+\tilde{\mathbf{K}}_{IJ}^{s} \tag{22}$$

Numerical examples and results

Firstly, the shear-locking test is presented. Isotropic simply supported square plate with side length $a=10$ under uniform load P. The deflection is normalized as $\hat{w}=w/(Pa^4/100D)$, where $D=Et^3/12(1-v^2)$. **Fig. 3** shows the locking test results for the isotropic clamped square plate. It can be seen that present method can eradicate the shear-locking.

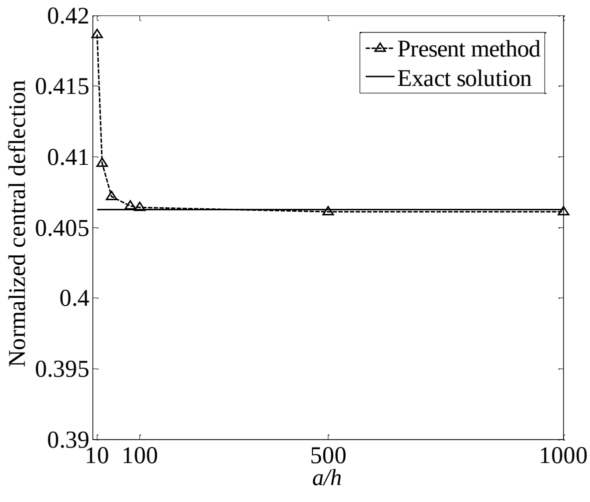
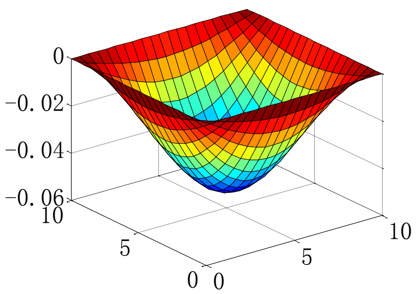


Fig. 3 Shear-locking test for a simply supported square plate

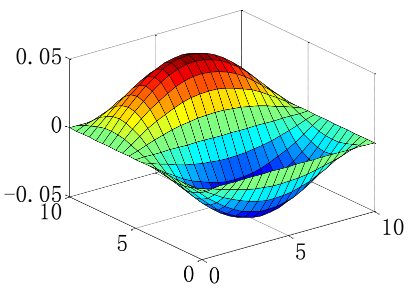
Secondly, two simply supported symmetric cross-ply ($0^\circ/90^\circ/0^\circ$) square plates are considered, the length to thickness ratios is $a/h = 100$, The material parameters are $\rho = 1643$, $E_{22} = 7.6\times10^9$, $E_{11} = 25E_{22}$, $G_{12} = G_{13} = 0.5E_{22}$, $G_{23} = 0.2E_{22}$, and $v_{11} = 0.25$. Both present results , Liu's and Reddy's solutions are listed in **Table 1** for comparison purpose. Good agreement can be observed for all modes. In addition, the first six modes are shown in **Fig. 4**.

Table 1. Nondimensionalized natural frequencies of simply supported symmetric cross-ply ($0^\circ/90^\circ/0^\circ$) square composite plates with RI ($\bar{\omega} = \omega a^2\sqrt{\rho E_{22}h^2}$)

Mode	Number of elements	$\bar{\omega}$		
		Present method	Liu GR et al.	Reddy
1	8×8	15.1795		
	16×16	15.1645	15.127	15.183
	20×20	15.1656		
	24×24	15.1665		
2	8×8	23.0857		
	16×16	22.8351	22.658	22.817
	20×20	22.8193		
	24×24	22.8132		
3	8×8	41.8609		
	16×16	40.3884	39.644	40.153
	20×20	40.2707		
	24×24	40.2198		
4	8×8	56.1309		
	16×16	55.9385	55.452	56.210
	20×20	55.9479		
	24×24	55.9575		
5	8×8	60.2757		
	16×16	59.9252	59.289	60.211
	20×20	59.9378		
	24×24	59.9510		



Mode 1



Mode 2

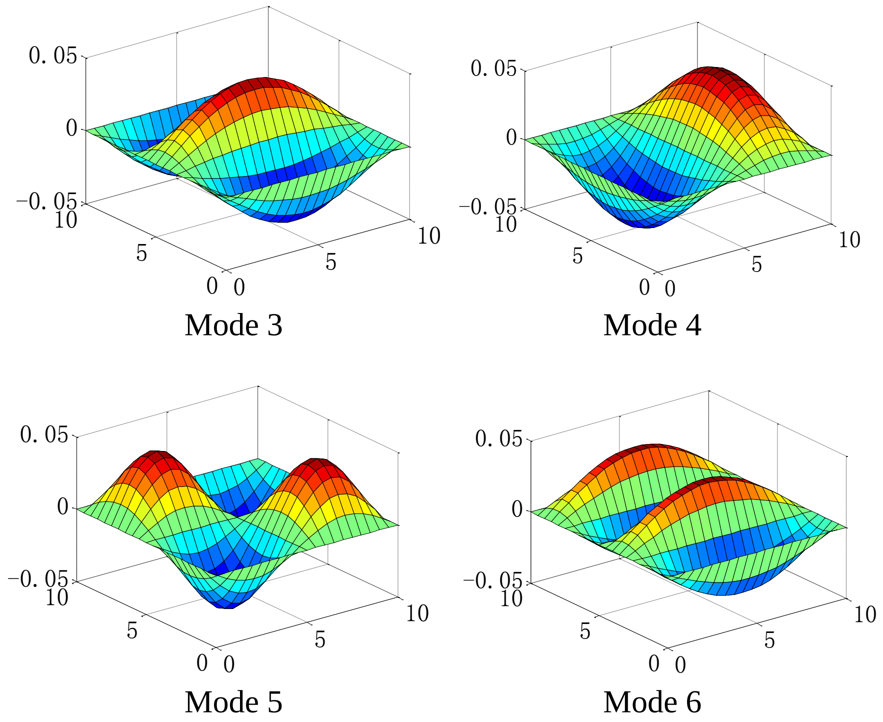


Fig. 4 The first six modes of the simply supported symmetric cross-ply square plates

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