Consistent meshfree method for phase-field model of brittle fracture

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Abstract

Numerical solution of the phase-field model of brittle fracture involves the capture of high gradients at cracks. Meshfree methods are convenient to construct high order approximation and to implement local refinement. This facilitates the capture of local high gradients to a large extent. Due to this reason, this paper introduces the element-free Galerkin (EFG) method, which is one of the major meshfree methods, to the phase-field modeling of crack growth. The meshfree discretization of the governing equations is described. To further improve the computational efficiency, the consistent integration scheme for quadratic EFG method is employed. The capability of the proposed consistent meshfree method for phase-field model of brittle fracture in predicting the load-displacement response and crack patterns is demonstrated by numerical examples.

Keywords: Meshfree; Phase-field model; Brittle fracture; EFG; Crack propagation

Introduction

The prediction of failure mechanisms due to crack initiation and propagation in solids is of great importance for engineering applications. The Griffith theory provides a criterion for crack propagation, but it is insufficient to determine curvilinear crack paths, crack kinking and branching angles. In particular, such a theory is unable to predict crack initiation. These defects of the classical Griffith-type theory of brittle fracture can be overcome by variational methods based on energy minimization as suggested by Francfort & Marigo [1], see also Bourdin, Francfort and Marigo [2]-[3]. The approximation regularizes a sharp crack surface topology in the solid by diffusive crack zones governed by a scalar auxiliary variable. Kuhn and Müller[4], using the framework of Gurtin [5] on the thermodynamics of order parameter based models, reformulated the energy minimization problem as the system of the stress equilibrium equation and a Ginzburg–Landau type evolution equation for phase field. However, the phase-field formulation does not distinguish between fracture behavior in tension and compression [3]. To avoid such situations, and, additionally, to prevent the interpenetration of the crack faces under compression, two modified regularized formulation were proposed by Amor et al. [6] and Miehe et al. [7]-[8] using an additive decomposition of the elastic energy density. Borden [9] presented a higher-order phase-field model formulation which could gain more regular and faster converging solutions of the variational problem of brittle fracture. Ambati et al. [10] and Nguyen [11] et al. modified the phase-field model in [8] in computational efficiency and accuracy, respectively. In order to avoid split the fourth order differential equation into two second order differential equations, Amiri et al. [12] applied a fourth order phase-field model for fracture based on local maximum entropy (LME) approximants. Now the phase field approaches to fracture have been applied in many fields such as the fracture in biological tissues, poroelastic medium and so on. They offer important new perspectives towards the theoretical and computational modeling of complex crack topologies. But these models have the high gradient of phase field at cracks which need high smoothness and accuracy and result in much more computational cost.

EFG method as a representative of Meshfree methods [13]-[16], has the advantages of high smoothness and offers substantial potential in many fields. However, their relatively low computational efficiency seriously hinders their applications. One main reason is that a plenty of integration points cost much more CPU time. Many efforts have been devoted to developing stable and efficient integration methods with reduced number of integration points [17]-[20]. Among them, the CEFG method by Duan [19] not only dramatically reduces the number of quadrature points in domain integration but also accurately passes the linear and quadratic patch tests, and remarkably improves the computational efficiency, accuracy and convergence of the standard EFG methods. Now it has been applied in many fields such as heat conduction [21], dynamics [22], elastic-plastic [23], nearly-incompressible elasticity [24]-[25] and so on. It is very promising.
In this paper, we will apply CEFG method to the phase-field model of brittle fracture, deduce the meshfree discretization of phase-field model and demonstrate its validity.

Phase-field model of brittle fracture

The variational approach to fracture mechanics provided by Francfort and Marigo [1] introduces the following energy functional for cracked body:

$$E(\mathbf{u},\Gamma)=E_u(\mathbf{u},\Gamma)+E_s(\Gamma)=\int_\Omega W_u(\boldsymbol{\varepsilon}(\mathbf{u}))d\Omega+\int_\Gamma g_c d\Gamma \tag{1}$$

where $E_u(\mathbf{u},\Gamma)$ represents the elastic energy stored in the cracked body, $E_S(\Gamma)$ is the energy required to create the crack according to the Griffith criterion, $\boldsymbol{\varepsilon}$ is the strain field, $\mathbf{u}$ is the displacement field, Γ is the geometry of the crack, W_u is the energy density function and g_c is the fracture toughness. In a regularized framework (phase field method), the above functional is substituted by the functional:

$$E(\mathbf{u},\Gamma)=\int_\Omega W_u(\boldsymbol{\varepsilon}(\mathbf{u}),d)d\Omega+g_c\int_\Omega \gamma(d)d\Omega \tag{2}$$

where

$$\gamma(d)=\frac{1}{2l}d^2+\frac{l}{2}\nabla d\bullet\nabla d \tag{3}$$

Bounded phase field $d\in(0,1)$, $d=0$ and $d=1$ correspond to the unbroken and fully broken states, respectively. l is the regularization parameter which is related to the diffusive approximation of the sharp crack. The total energy is then rewritten as $E=\int_\Omega Wd\Omega$ in which

$$W=W_u(\boldsymbol{\varepsilon}(\mathbf{u}),d)+g_c\gamma(d) \tag{4}$$

can be identified as the free energy.
In order to distinguish between fracture behavior in tension and compression, assuming isotropic elastic behavior of the body accounting for damage induced by traction only, we choose the following form for W_u

$$W_u(\boldsymbol{\varepsilon}(\mathbf{u}),d)=\Psi^+(\boldsymbol{\varepsilon})[g(d)+k]+\Psi^-(\boldsymbol{\varepsilon}) \tag{5}$$

k is a numerical parameter. The strain field is decomposed into tensile and compressive sections as

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^++\boldsymbol{\varepsilon}^- \tag{6}$$

and

$$\Psi^\pm(\boldsymbol{\varepsilon})=\frac{1}{2}\lambda\langle tr(\boldsymbol{\varepsilon})\rangle_\pm^2+\mu tr[(\boldsymbol{\varepsilon}^\pm)^2] \tag{7}$$

where λ , μ are the elastic bulk modulus and shear modulus.

$$\boldsymbol{\varepsilon}^\pm=\sum_{I=1}^{D}\langle\varepsilon_I\rangle_\pm\mathbf{n}_I\otimes\mathbf{n}_I \tag{8}$$

ε_I and $\mathbf{n}_I$ are the eigenvalues and eigenvectors of $\boldsymbol{\varepsilon}$. In (8) $\langle x \rangle_{\pm} = \frac{1}{2}(x \pm |x|)$. The degradation function $g(d) = (1-d)^2$.

To handle the irreversibility of the crack phase-field evolution in a general and possibly cyclic, loading/unloading scenario. Miehe et al. [8] introduced the strain history functional:

$$H(\mathbf{x},t) = \max_{\tau \in [0,t]} \Psi^{+}(\boldsymbol{\varepsilon}(\mathbf{x},\tau)) \tag{9}$$

For a discussion and justification of the use of this function, the reader is invited to refer to the mentioned reference. Then we can obtain the governing equations for phase field and displacement field as follows

$$\begin{cases} div\boldsymbol{\sigma}(\mathbf{u},d) = \mathbf{0} \\ 2(1-d)H + \dfrac{g_c}{l}\left(d - l^2 \Delta d\right) = 0 \end{cases} \tag{10}$$

in the domain Ω along with boundary conditions:

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on} \quad \partial\Omega_{\sigma} \tag{11}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \partial\Omega_{u} \tag{12}$$

$$d(\mathbf{x}) = 1 \quad \text{on} \quad \Gamma \tag{13}$$

$$\nabla d(\mathbf{x}) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \tag{14}$$

where

$$\boldsymbol{\sigma}(\mathbf{u},d) = \left((1-d)^2 + k\right)\left\{\lambda Tr\boldsymbol{\varepsilon}_{+} 1 + 2\mu\boldsymbol{\varepsilon}^{+}\right\} + \lambda Tr\boldsymbol{\varepsilon}_{-} 1 + 2\mu\boldsymbol{\varepsilon}^{-} \tag{15}$$

Meshfree discretization

The phase field and phase field gradient are approximated

$$d(\mathbf{x}) = \sum_I \mathbf{N}_d^I(\mathbf{x}) d_I \quad \nabla d(\mathbf{x}) = \sum_I \mathbf{B}_d^I(\mathbf{x}) d_I \tag{16}$$

$\mathbf{N}_d(\mathbf{x})$ and $\mathbf{B}_d(\mathbf{x})$ are vectors and matrices of meshfree nodal shape functions and their derivatives for phase field, respectively. By using the standard Galerkin procedure with Penalty method enforcing the essential boundary condition for initial crack induced by phase-field, the final discretized equation is

$$(\beta \mathbf{K}_d^p + \mathbf{K}_d)\mathbf{d} = \beta \mathbf{F}_d^p + \mathbf{F}_d \tag{17}$$

where β is the penalty parameter.

$$\mathbf{K}_d^p = \int_{\Gamma} \mathbf{N}_d^{\mathrm{T}} \mathbf{N}_d d\Gamma, \mathbf{F}_d^p = \int_{\Gamma} \mathbf{N}_d^{\mathrm{T}} d d\Gamma \tag{18}$$

$$\mathbf{K}_d = \int_{\Omega} \left\{ \left(2H + \frac{g_c}{l}\right) \mathbf{N}_d^{\mathrm{T}} \mathbf{N}_d + g_c l \mathbf{B}_d^{\mathrm{T}} \mathbf{B}_d \right\} d\Omega \tag{19}$$

$$\mathbf{F}_d = \int_{\Omega} 2\mathbf{N}_d^{\mathrm{T}} H d\Omega \tag{20}$$

The displacement field is approximated

$$\mathbf{u}(\mathbf{x}) = \sum_I \mathbf{N}_I(\mathbf{x}) \mathbf{u}_I \tag{21}$$

$\mathbf{N}(\mathbf{x})$ is the matrices of meshfree nodal shape functions of displacement field. Introduce two shifted strain matrix

$$\boldsymbol{\varepsilon}^{+} = \mathbf{P}^{+}\boldsymbol{\varepsilon}, \ \boldsymbol{\varepsilon}^{-} = \mathbf{P}^{-}\boldsymbol{\varepsilon} \tag{22}$$

By using the standard Galerkin procedure with Nitsche's Method [26] enforcing the essential boundary condition, the final discretized equation is

$$\left\{\mathbf{K}_u - \mathbf{K}^{\partial\Omega_u} + \beta\mathbf{K}_u^p\right\}\mathbf{u} = \mathbf{F}_u - \mathbf{F}^{\partial\Omega_u} + \beta\mathbf{F}_u^p \tag{23}$$

where

$$\mathbf{K}_u = \int_\Omega \mathbf{B}^{\mathrm{T}}\mathbf{D}(d)\mathbf{B}\mathrm{d}\Omega \qquad \mathbf{F}_u = \int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{f}d\Omega + \int_{\partial\Omega_\sigma} \mathbf{N}^{\mathrm{T}}\bar{\mathbf{t}}d\partial\Omega_\sigma \qquad \mathbf{B} = \mathbf{L}\mathbf{N} \tag{24}$$

$$\mathbf{K}^{\partial\Omega_u} = \int_{\partial\Omega_u}\left[\mathbf{N}^{\mathrm{T}}\left(\mathbf{L}_n^{\mathrm{T}}\mathbf{D}(d)\mathbf{B}\right) + \left(\mathbf{L}_n^{\mathrm{T}}\mathbf{D}(d)\mathbf{B}\right)^{\mathrm{T}}\mathbf{N}\right]\mathrm{d}\partial\Omega_u \tag{25}$$

$$\mathbf{F}^{\partial\Omega_u} = \int_{\partial\Omega_u}\left(\mathbf{L}_n^{\mathrm{T}}\mathbf{D}(d)\mathbf{B}\right)^{\mathrm{T}}\bar{\mathbf{u}}\mathrm{d}\partial\Omega_u \tag{26}$$

$$\mathbf{K}_u^p = \int_{\partial\Omega_u}\mathbf{N}^{\mathrm{T}}\mathbf{N}\mathrm{d}\partial\Omega_u,\ \ \mathbf{F}_u^p = \int_{\partial\Omega_u}\mathbf{N}^{\mathrm{T}}\bar{\mathbf{u}}\mathrm{d}\partial\Omega_u \tag{27}$$

$$\mathbf{L} = \begin{bmatrix}\partial/\partial x & 0 & \partial/\partial y\\ 0 & \partial/\partial y & \partial/\partial x\end{bmatrix}^{\mathrm{T}} \qquad \mathbf{L} = \begin{bmatrix} n_x & 0 & n_y\\ 0 & n_y & n_x\end{bmatrix}^{\mathrm{T}} \tag{28}$$

$$\mathbf{D}(d) = \left((1-d)^2 + k\right)\left\{\lambda \mathrm{R}^+\mathbf{I}\mathbf{I}^{\mathrm{T}} + 2\mu\mathbf{P}^+\right\} + \lambda\mathrm{R}^-\mathbf{I}\mathbf{I}^{\mathrm{T}} + 2\mu\mathbf{P}^- \tag{29}$$

$$\mathbf{I} = [1,1,0]^{\mathrm{T}},\ R^+ = \frac{1}{2}\left(sign(tr(\boldsymbol{\varepsilon}))+1\right),\ R^- = \frac{1}{2}\left(sign(-tr(\boldsymbol{\varepsilon}))+1\right) \tag{30}$$

In this paper, the meshfree nodal shape functions and their derivatives for phase field and displacement field are identical. The solution of them can be obtained in [13] and [19]. We adopted the modified derivatives of meshfree nodal shape functions in [19], namely CEFG.

Results

Single edge notched pure shear test

Consider a squared plate containing horizontal notch located at middle height from the left edge with a length of 0.5 mm. The geometric setup is depicted in Fig. 1. In order to capture the crack pattern properly, the mesh is refined in areas where the crack is expected to propagate. An effective nodal distance h≈0.001 mm in the critical zone is obtained. The regularization parameter l = 3h. The elastic bulk modulus is chosen to λ =121.15 kN/mm^2, the shear modulus to μ=80.77 kN/mm^2 and the critical energy release rate to g_c=2.7×10^{-3} kN/mm.

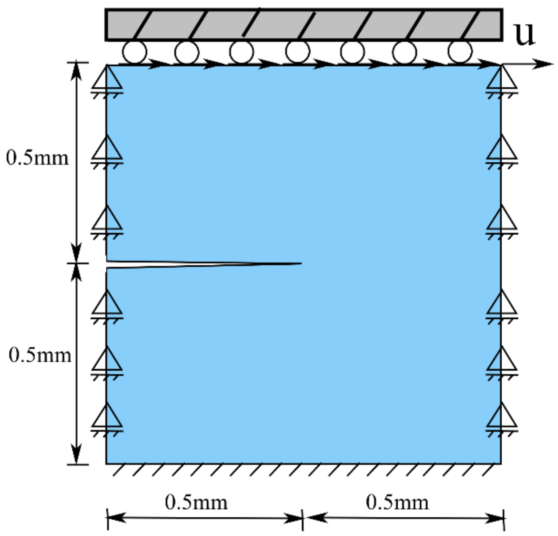

Fig. 1.Geometry and boundary conditions of single edge notched pure shear test

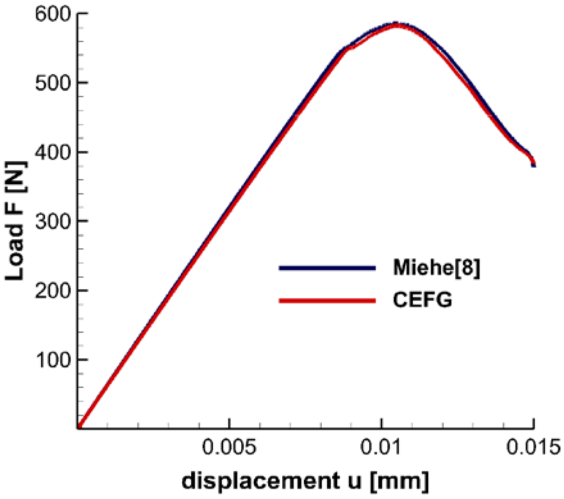

Fig. 2. Load-deflection curves of single edge notched pure shear test

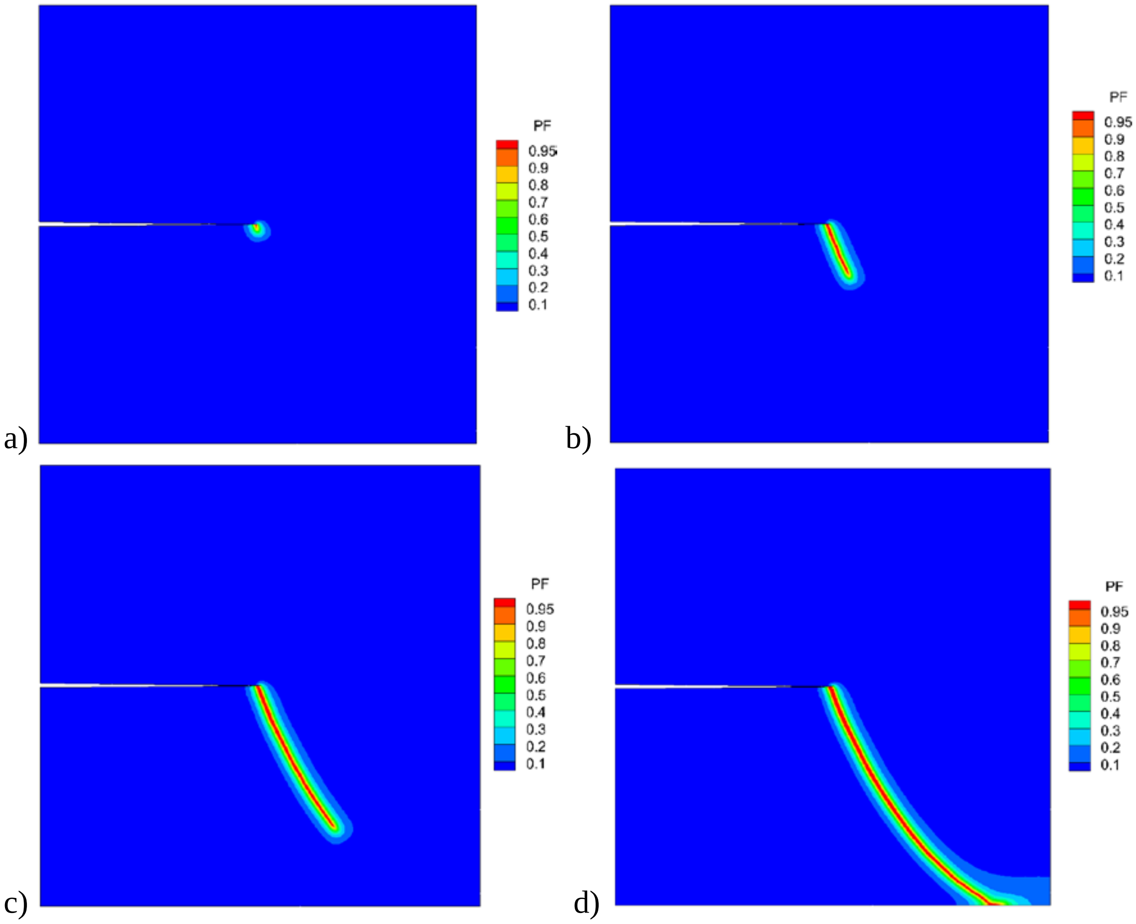

Fig. 3. Crack pattern of single edge notched pure shear test at a displacement of a) $u=9.0\times10^{-3}$ mm, b) $u=11.0\times10^{-3}$ mm, c) $u=13\times10^{-3}$ mm, d) $u=15\times10^{-3}$ mm

The computation is performed in a monotonic displacement driven context with constant increment $\Delta u=1\times10^{-5}$ mm. The load-deflection curves are depicted in Fig. 2. Obviously, it agrees well with the result in Miehe [8]. Figure 3 shows the crack patterns at several stages of loading. Here the red and blue colors indicate the damaged and undamaged material, respectively. From the Fig.3, we can find the phase field at crack is very smooth and CEFG method can accurately capture the crack pattern of single edge notched pure shear test.

Symmetric three point bending test

This test is a simply supported notched beam. The geometric setup as well as the loading conditions are illustrated in Fig. 4. The discretization is refined in the expected crack propagation zone. And an effective nodal distance is h≈0.008 mm in the critical zone. The bulk modulus is chosen to λ=12.00 kN/mm^2, the shear modulus to μ=8.0 kN/mm^2 and the critical energy release rate to g_c=5.0×10^{-4} kN/mm.

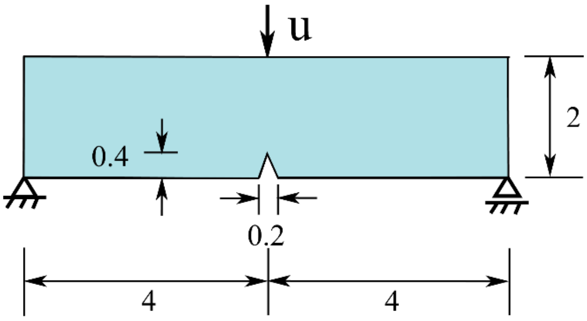

Fig. 4.Geometry and boundary conditions of simply supported notched beam

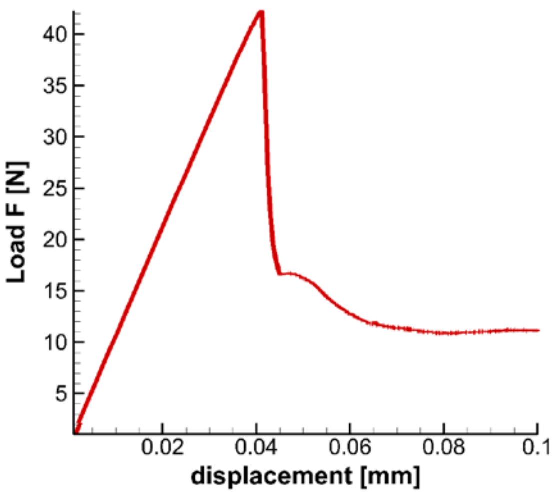

Fig. 5. Load-deflection curves of simply supported notched beam

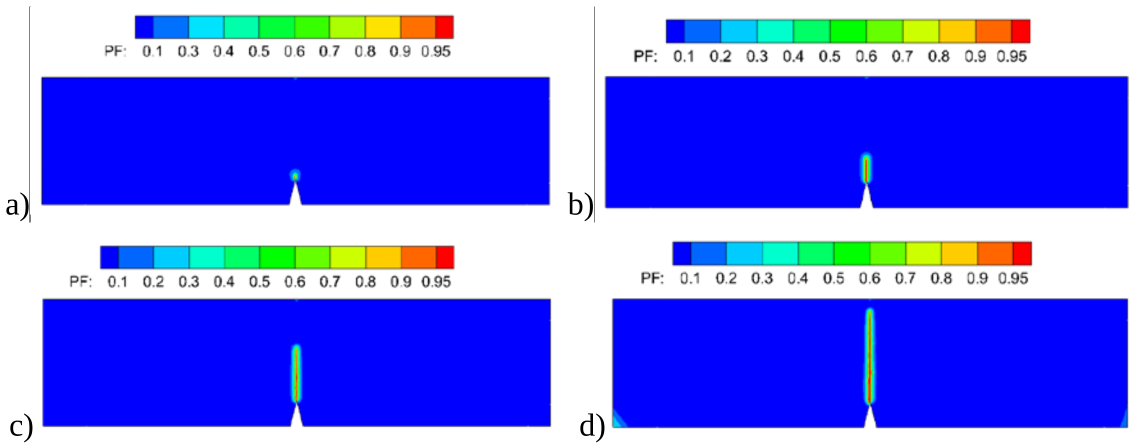

Fig. 6. Crack pattern of simply supported notched beam at a displacement of a) u=3.9×10^{-2} mm, b) u=4.131×10^{-2} mm, c) u=4.5×10^{-2} mm, d) u=1×10^{-1} mm

The computation is performed in a monotonic displacement driven context with constant increment Δu=1×10^{-3} mm in the first 39 loading steps. Continuing crack propagation then demands for an adjustment of the displacement increment to $\Delta u=1\times10^{-5}$ mm for the subsequent 600 loading steps and $\Delta u=1\times10^{-4}$ for the last 550 loading steps. The load-deflection curves obtained are depicted in Fig. 5. It shows CEFG method can catch the basic characteristics of brittle fracture. The resulting contour plots of the crack topology are given in Fig. 6. We can reach the same conclusion like single edge notched pure shear test.

Conclusions

The quadratic CEFG method for the phase-field model of brittle fracture is implemented. The feasibility of the proposed method for modeling crack extension is investigated by two numerical examples, i.e. the single edge notched pure shear test and the simply supported notched beam. It is demonstrated that the load-displacement response and crack patterns are predicted correctly by the proposed consistent meshfree method.

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