An Optimal Control Obtained by Finite Dimensional Approximation for a Flexible Robot Arm

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Abstract
In this paper, we are concerned with a flexible robot arm formulated by partial differential equations with initial and boundary conditions. An optimal energy control of the robot arm is investigated after the system has been transformed to an abstract evolution system in an appropriate Hilbert space. An optimal energy control problem is discussed, and evantually, it is shown that an optimal energy control can be obtained by a finite dimensional approximation.

Keywords
Flexible Robot Arm, Optimal Control, Finite Dimensional Approximations.

1 Introduction

The vibration suppression of flexible Euler-Bernoulli beams has been studied extensively in many articles [1]−[7] due to its wide applications. We have studied control theory for flexible robot arms in the articles [8−11]. In the present paper, we are going to discuss a kind of optimal energy control problems for a flexible robot system.

We are now concerned with a flexible robot arm in the x-y plane. Since any motion of the robot arm in the $x-y$ plane can be decomposed into its x and y components, the vibration in the x-direction and y-direction can be considered independently which can be formulated by the following partial differential equations with the initial and the boundary conditions [9]:

$$\rho\ddot{w}(t,r) + EIw''''(t,r) = -\rho\ddot{x}(t), \qquad 0 < r < t, \tag{1.1}$$

$$w(t,0) = w'(t,0) = 0, \tag{1.2}$$

$$M[\ddot{w}(t,l) + \ddot{x}(t)] - EIw'''(t,l) = 0, \tag{1.3}$$

$$J\ddot{w}'(t,l) + EIw''(t,l) = 0, \tag{1.4}$$

$$w(0,r) = w_0(r), \quad \dot{w}(0,r) = w_1(r), \tag{1.5}$$

where m is the mass of a moving body driven by a control motor, M is the mass of the payload of the flexible robot arm that is attached to this moving body, $w(t,r)$ represents the amplitude of vibration of the flexible beam at time t and position r, $\ddot{w}(t)$ denotes the acceleration of the moving body in the x-direction, "." denotes the time derivative, and "′" denotes the spatial derivative, ρ denotes the line density of mass for the arm, EI denotes the bending rigid degree of the flexible beam, l denotes the length of the arm, J denotes the turning inertia, $w_0(r)$ and $w_1(r)$ denote the initial displacement and initial velocity of the arm, respectively.

For the motor system, we shall establish the following control equation:

$$m\ddot{x}(t) = u(t) - EIw'''(t,0), \tag{1.6}$$

where the sliding friction was neglected and $u(t)$ is a control.
Let $y(t,r)$ be the total displacement in the x-direction of the flexible beam. Thus, we have

$$y(t,r) = w(t,r) + x(t). \tag{1.7}$$

Substituting (1.7) into the system (1.1)-(1.5) yields the following controlled closed-loop system equation about state $y(t,r)$:

$$\begin{aligned}
\rho\ddot{y}(t,r)+EIy''''(t,r) &= 0, \quad 0<r<l, t>0, &(1.8)\\
y'(t,0) &= 0, &(1.9)\\
m\ddot{y}(t,0)+EIy'''(t,0) &= u(t), &(1.10)\\
M\ddot{y}(t,l)-EIy'''(t,l) &= 0, &(1.11)\\
J\ddot{y}'(t,l)+EIy''(t,l) &= 0. &(1.12)
\end{aligned}$$

In order to investigate the system (1.8)-(1.12) under the abstract frame, we now consider a real Hilbert space $H = R^3 \times L^2_\rho(0,l)$ equipped with the inner product as

$$(\Phi_1,\Phi_2)_H = m\xi_1\xi_2 + M\eta_1\eta_2 + J\zeta_1\zeta_2 + \langle\varphi_1,\varphi_2\rangle_\rho$$

where $\Phi_i = [\xi_i,\eta_i,\zeta_i,\varphi_i]^\tau \in H$, $i=1,2$, $\langle\varphi_1,\varphi_2\rangle_\rho = \int_0^l \rho\varphi_1(x)\overline{\varphi_2}(x)dx$, and τ means the transpose. We define a linear operator A with domain $D(A)$ in H as follows:

$$A\tilde{\varphi} = \begin{bmatrix} \frac{EI}{m}\varphi'''(0) \\ -\frac{EI}{M}\varphi'''(l) \\ \frac{EI}{J}\varphi''(l) \\ \frac{EI}{\rho}\varphi''''(\cdot) \end{bmatrix}, \quad \text{for} \quad \tilde{\varphi} = \begin{bmatrix} \varphi(0) \\ \varphi(l) \\ \varphi'(l) \\ \varphi(\cdot) \end{bmatrix} \in D(A)$$

where $D(A) = \{\tilde{\varphi} \in H : \varphi,\varphi',\varphi'',\varphi''',\varphi'''' \in L^2_\rho(0,l), \varphi' = 0\}$.
Using the operator A, the system (1.8)-(1.12) becomes the following second-order abstract evolution equation in H:

$$\frac{d^2\tilde{y}(t)}{dt^2} + A\tilde{y}(t) = bu(t), \tag{1.13}$$

where $\tilde{y}(t) = [y(t,0), y(t,l), y'(t,l), y(t,\cdot)]^\tau, b = [1/m,0,0,0]^\tau$.
First, we are going to propose and prove the following theorems[9]

Theorem 1.1 *$A : D(A) \to H$ is a nonnegative self-adjoint operator.*

Theorem 1.2 *The resolvent of A is a compact operator.*

Theorem 1.3 *The spectrum of A consists of only nonnegative eigenvalues with single multiplicity.*

We now define a Hilbert space $\mathcal{H} = H \times H$, and a linear operator $\mathcal{A}$ on $\mathcal{H}$ as follows

$$\mathcal{A} = \begin{bmatrix} 0 & A_r^{1/2} \\ -A_r^{1/2} + rA_r^{-1/2} & 0 \end{bmatrix}$$

where r is in $\rho(A)$, the resolvent set of A, and $D(\mathcal{A}) = D(A_r^{1/2}) \times D(A_r^{1/2})$. We also denote $[0,b]^\tau$ by $\mathcal{B}$, where b is defined in (1.13). Let us consider a subspace $\mathcal{T}$ of $\mathcal{H}$ consisting of $z = [z_1,z_2]^\tau$, where $z_1 = A_r^{1/2}\tilde{y}$, and $z_2 = \dot{\tilde{y}}$, and $\tilde{y}$ is defined in (1.13). In these notations, the equation (1.13) with initial conditions

$$\begin{aligned}
\tilde{y}(0) &= \tilde{y}_0 &(1.14)\\
\dot{\tilde{y}} &= \tilde{y}_1 &(1.15)
\end{aligned}$$

becomes a first-order evolution equation in $\mathcal{T}$ with initial conditions as follows:

$$\begin{aligned}
\frac{dz}{dt} &= \mathcal{A}z + \mathcal{B}u, &(1.16)\\
z_0 &= [A_r^{1/2}\tilde{y}_0, \tilde{y}_1]^\tau. &(1.17)
\end{aligned}$$

The semigroup property of the linear operator $\mathcal{A}$ has been proved in the article [9] as follows:

Theorem 1.4 *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0 − semigroup $T(t)$, $t \geq 0$.*

2 A Minimum Energy Problem

Since $T(t)$ is the semigroup of linear operators generated by the operator $\mathcal{A}$ (See the Theorem 1.4), it follows from the theory of semigroup of linear operators that the system (1.16)- (1.17) has an unique mild solution [12] given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)\mathcal{B}u(s)\,ds \tag{2.18}$$

Let $\varphi(\cdot)$ be an arbitrary element in $C([0,T];\mathcal{H})$, and

$$\rho = inf_{u\in L^2([0,T];\mathcal{H})}\|\varphi(t) - T(t)z_0 - \int_0^t T(t-s)\mathcal{B}u(s)\,ds\|\,, \tag{2.19}$$

define the admissible control set of the system (1.16)-(1.17)as follows

$$\begin{aligned} U_{ad} = \{u \in L^2([0,T];\mathcal{H}) : \|\varphi(t) - T(t)z_0 \\ - \int_0^t T(t-s)\mathcal{B}u(s)\,ds\| \le \rho + \epsilon\} \end{aligned} \tag{2.20}$$

where ϵ is any positive number.
It can be seen from (2.2) that U_{ad} is not empty and contains infinitely many elements related to φ and ϵ. The minimum energy control problem is actually to find the element u, satisfying

$$\|u_0\| = min\{\|u\| : u \in U_{ad}\} \tag{2.21}$$

where u_0 is said to be a minimum energy control element.

Lemma 2.1 *The admissible control set U_{ad} defined by (2.3) is a closed convex set in Hilbert space $L^2([0,T];\mathcal{H})$.*

Proof. Convexity. For any u_1, $u_2 \in U_{ad}$ and a real number λ, $0 < \lambda < 1$, it is easy to see from (2.2) that

$$\|\varphi(t) - T(t)z_0 - \int_0^t T(t-s)\mathcal{B}u_i(s)\| \quad \le \quad \rho + \epsilon \quad i = 1, 2$$

and hence

$$\begin{aligned} & \|\varphi(t) - T(t)z_0 - \int_0^t T(t-s)\mathcal{B}[\lambda u_1(s) \\ & \qquad + (1-\lambda)u_2(s)]\,ds\|x \\ = \; & \|\varphi(t) - T(t)z_0 - \int_0^T T(t-s)[\lambda\mathcal{B}u_1(s) \\ & \qquad + (1-\lambda)\mathcal{B}u_2(s)]\,ds\| \\ \le \; & \lambda\|\varphi(t) - T(t)z_0 - \int_0^t T(t-s)\mathcal{B}u_1(s)\,ds\| \\ & + (1-\lambda)\|\varphi(t) - T(t)z_0 - \int_0^t T(t-s)\mathcal{B}u_2(s)\,ds\| \\ \le \; & \lambda(\rho+\epsilon) + (1-\lambda)(\rho+\epsilon) = \rho + \epsilon. \end{aligned}$$

Since $\lambda u_1 + (1-\lambda)u_2 \in L^2([0,T];H)$, it follows that $\lambda u_1 + (1-\lambda)u_2 \in U_{ad}$, this implies that U_{ad} is a convex subset of $L^2([0,T];H)$.
Closedness. Suppose $\{z_n\} \subset U_{ad}$, and $\lim_{n\to\infty}\|u_n - u^*\| = 0$. It can be shown that $u^* \in U_{ad}$. In fact, we see from the definition of U_{ad} that

$$\|\varphi(t) - T(t)z_0 - \int_0^t T(t-s)\mathcal{B}u_n(s)\,ds\| \le \rho + \epsilon, \quad n = 1, 2, \cdots$$

Since $T(t),\ t \geq 0$ is a C_0-semigroup in Hilbert space $\mathcal{H}$, there is a constant $M > 0$ such that $\sup_{0\leq t\leq T} \|T(t)\| \leq M$. On the other hand, since $u(s)$ is differentiable on $[0, T]$, it is continuous on $[0, T]$, and hence $\{u(s) : s \in [0, T]\}$ is a bounded set in $L^2([0, T]; Y)$. Thus there is a constant $N > 0$ such that $\|Bu(s)\| \leq N\ \ (0 \leq s \leq T)$ and

$$\begin{aligned} & \|\varphi(t) - T(t)z_0 - \int_0^t T(t-s)\mathcal{B}u^*(s)\ ds\| \\ \leq\ & \|\varphi(t) - T(t)z_0 - \int_0^t \mathcal{B}u_n(s)\ ds\| \\ & \qquad + \|\int_0^t T(t-s)\mathcal{B}[u_n(s) - u^*(s)]\ ds\| \\ \leq\ & \rho + \epsilon + M\|u_n - u^*\| \cdot NT \end{aligned}$$

Letting $n \to \infty$ leads to

$$\|\varphi(t) - T(t)z_0 - \int_0^t T(t-s)\mathcal{B}u^*(s)\ ds\| \leq \rho + \epsilon\,.$$

Thus, $u^* \in U_{ad}$, and U_{ad} is a closed set and the proof is complete.

3 Optimal Energy Control of the System

In this section, we will discuss the existence and uniqueness of optimal energy control for the flexible robot system (1.16)-(1.17). Let us begin with the following theorem.

Theorem 3.1 *There exists an unique minimum energy control element in the admissible control set U_{ad} corresponding to the system (1.16) and (1.17).*

Proof. Since $L^2([0, T], Y)$ is a Hilbert space, it is naturally a strict convex Banach Space. From the preceding Lemma 2.1, we have seen that U_{ad} is a closed convex set in $L^2([0, T], Y)$, and it follows from [13] that there is an unique element $u_0 \in U_{ad}$ such that

$$\|u_0\| = min\ \ \{\|u\| : u \in U_{ad}\}$$

According to the definition (2.4), u_0 is just the desired minimum energy control element of the system (1.16)-(1.17). The proof is complete.

Finally, we shall show that the minimum energy control element can be approached.

Theorem 3.2 *Suppose that u_0 is the minimum energy control element of the system (1.16)-(1.17), then there exists a sequence $\{u_n\} \subset U_{ad}$ such that $\{u_n\}$ converges strongly to u_0 in $L^2([0, T]; Y)$, namely,*

$$\lim_{n\to\infty} \|u_n - u_0\| = 0$$

Proof. Let $\{u_n\}$ be a minimized sequence in the admissible control set U_{ad}, then it follows that

$$\|u_{n+1}\| \leq \|u_n\|, \quad n = 1, 2, \cdots \tag{3.22}$$

and

$$\lim_{n\to\infty} \|u_n\| = inf\{\|u\| : u \in U_{ad}\} \tag{3.23}$$

It is obvious that $\{u_n\}$ is a bounded sequence in $L^2([0, T]; Y)$, and so there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to an element $\widetilde{u}$ in $L^2([0, T]; Y)$[14].

Since U_{ad} is a closed convex set in $L^2([0, T]; Y)$ based on the Lemma 2.1, we see from Mazur's Theorem that U_{ad} is a weakly closed set in $L^2([0, T]; Y)$, thus $\tilde{u} \in U_{ad}$. Combining (3.2) and employing the properties of limits of weakly convergent sequence on norm yield

$$\begin{aligned} inf\{\|u\| : u \in U_{ad}\} \leq \|\tilde{u}\| \quad & \leq \quad \underline{\lim}_{k\to\infty} \|u_{n_k}\| \\ = \lim_{n_k\to\infty} \|u_{n_k}\| = \lim_{n\to\infty} \|u_n\| \quad & = \quad \inf\{\|u\|; u \in U_{ad}\}. \end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \|u_n\| = \|\tilde{u}\| \tag{3.24}$$

and

$$\|\tilde{u}\| = inf\{\|u\| u \in U_{ad}\}. \tag{3.25}$$

Since $\{u_{n_k}\}$ is weakly convergent to $\tilde{u}$, it follows from (3.3) that $\{u_{n_k}\}$ converges to $\tilde{u}$. Therefore, we see from the Theorem 3.1 and (3.4) that $\tilde{u} = u_0$, namely, $\tilde{u}$ is the minimum energy control element. Thus, $\{u_{n_k}\}$ strongly converges to the minimum energy control element in $L^2([0,T];Y)$. Without loss of generality, we can rewrite $\{u_{n_k}\}$ by $\{u_n\}$, and the conclusion of theorem is now obtained.

4 Coclusion

In this paper, we have investigated a kind of optimal energy control for a flexible robot arm formulated by partial differential equations with initial and boundary conditions. After a discussion of minimum energy problem for the beam system, we have proposed and proved the existence and uniqueness Theorem 3.1 of the optimal energy control in terms of semigroup approach of linear operators. Finally, we gave an approximation result Theorem 3.2 that points out that the minimum energy control element can be approached by a weakly convergent sequence in the control space, and provides the theoretical basis of approximate computation for finding an optimal energy control element.

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