Rectangle clamped at one end: Exact solution

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Abstract

In the paper the exact solution of a boundary value problem for a rectangle clamped at one end is constructed. The solution is given in the form of explicit expansions in Papkovich–Fadle eigenfunctions. The coefficients of the expansions are clearly determined by means of functions biorthogonal to Papkovich–Fadle eigenfunctions.

Keywords: Plate; clamp; Papkovich–Fadle eigenfunctions; exact solution

Introduction

Numerous publications are devoted to the approximate and numerical solutions of boundary value problems of the theory of elasticity for a rectangle with a clamped end (ends). The main reason for special interest in these problems is, partly, that sometimes the obtained solutions of the same problem could noticeably differ in different authors' works, depending on the way which method (approach) was used for their construction. For example, at the angular point some solutions had a singularity that is characteristic of an infinite rectangular wedge, one face of which is rigidly clamped and to the other one is applied an external load. In other solutions this singularity was absent.

In this paper is constructed the exact solution of a boundary value problem of the theory of elasticity for a rectangle, the left end of which is clamped and on the right end is applied a normal load (even-symmetric and odd-symmetric deformations).

Statement of the Problem

Let us consider a rectangle $\{P: |y| \le 1,\ 0 \le x \le d\}$. We will assume that the long sides $y = \pm 1$ are free, i.e.

$$\sigma_y(x, \pm 1) = \tau_{xy}(x, \pm 1) = 0, \tag{1}$$

the left end $x = 0$ is clamped, and a normal load is applied on the right end $x = d$, i.e.

$$\begin{gathered} u(0, y) = v(0, y) = 0, \\ \sigma_x(d, y) = \sigma(y), \tau_{xy}(d, y) = 0. \end{gathered} \tag{2}$$

Then the solution in the rectangle written as expansions in Papkovich–Fadle eigenfunctions has the following form:

$$U(x, y) = \sum_{k=1}^{\infty} A_k \xi(\lambda_k, y) \sinh \lambda_k x + B_k \xi(\lambda_k, y) \cosh \lambda_k x + \\ + \overline{A_k} \xi(\overline{\lambda_k}, y) \sinh \overline{\lambda_k} x + \overline{B_k} \xi(\overline{\lambda_k}, y) \cosh \overline{\lambda_k} x,$$

$$V(x, y) = \sum_{k=1}^{\infty} A_k \chi(\lambda_k, y) \cosh \lambda_k x + B_k \chi(\lambda_k, y) \sinh \lambda_k x + \\ + \overline{A_k} \chi(\overline{\lambda_k}, y) \cosh \overline{\lambda_k} x + \overline{B_k} \chi(\overline{\lambda_k}, y) \sinh \overline{\lambda_k} x,$$

$$\sigma_x(x,y)=\sum_{k=1}^{\infty}A_k s_x(\lambda_k,y)\cosh\lambda_k x+B_k s_x(\lambda_k,y)\sinh\lambda_k x+ \\ +\overline{A_k}s_x\left(\overline{\lambda_k},y\right)\cosh\overline{\lambda_k}x+\overline{B_k}s_x\left(\overline{\lambda_k},y\right)\sinh\overline{\lambda_k}x, \tag{3}$$

$$\sigma_y(x,y)=\sum_{k=1}^{\infty}A_k s_y(\lambda_k,y)\cosh\lambda_k x+B_k s_y(\lambda_k,y)\sinh\lambda_k x+ \\ +\overline{A_k}s_y\left(\overline{\lambda_k},y\right)\cosh\overline{\lambda_k}x+\overline{B_k}s_y\left(\overline{\lambda_k},y\right)\sinh\overline{\lambda_k}x,$$

$$\tau_{xy}(x,y)=\sum_{k=1}^{\infty}A_k t_{xy}(\lambda_k,y)\sinh\lambda_k x+B_k t_{xy}(\lambda_k,y)\cosh\lambda_k x+ \\ +\overline{A_k}t_{xy}\left(\overline{\lambda_k},y\right)\sinh\overline{\lambda_k}x+\overline{B_k}t_{xy}\left(\overline{\lambda_k},y\right)\cosh\overline{\lambda_k}x.$$

Here $U(x,y)=Gu(x,y)$, $V(x,y)=Gv(x,y)$; $u(x,y)$ and $v(x,y)$ are displacements along the x-axis (longitudinal) and along the y-axis (transverse) respectively; G is the shear modulus; ν is Poisson's ratio.

Assume that the elementary part of the solution is already known. Satisfying the boundary conditions of (2) on the ends of the rectangle, we come to the problem of determining the coefficients a_k from the expansions

$$0=\sum_{k=1}^{\infty}B_k\xi(\lambda_k,y)+\overline{B_k}\xi\left(\overline{\lambda_k},y\right),$$

$$0=\sum_{k=1}^{\infty}A_k\chi(\lambda_k,y)+\overline{A_k}\chi\left(\overline{\lambda_k},y\right),$$

$$\sigma(y)=\sum_{k=1}^{\infty}A_k s_x(\lambda_k,y)\cosh\lambda_k d+B_k s_x(\lambda_k,y)\sinh\lambda_k d+ \\ +\overline{A_k}s_x\left(\overline{\lambda_k},y\right)\cosh\overline{\lambda_k}d+\overline{B_k}s_x\left(\overline{\lambda_k},y\right)\sinh\overline{\lambda_k}d, \tag{4}$$

$$0=\sum_{k=1}^{\infty}A_k t_{xy}(\lambda_k,y)\sinh\lambda_k d+B_k t_{xy}(\lambda_k,y)\cosh\lambda_k d+ \\ +\overline{A_k}t_{xy}\left(\overline{\lambda_k},y\right)\sinh\overline{\lambda_k}d+\overline{B_k}t_{xy}\left(\overline{\lambda_k},y\right)\cosh\overline{\lambda_k}d.$$

Following the general scheme of solving a boundary value problem for a half-strip [1, 2], with the help of the functions $u_k(y), v_k(y), x_k(y), t_k(y)$ biorthogonal to the Papkovich–Fadle eigenfunctions $\xi(\lambda_k,y)$, $\chi(\lambda_k,y)$, $s_x(\lambda_k,y)$, $s_y(\lambda_k,y)$, we obtain the system of algebraic equations for each $k\geq 1$:

$$0=B_k\lambda_k M_k+\overline{B_k\lambda_k M_k},$$

$$0=A_k M_k+\overline{A_k M_k},$$

$$\sigma_k^*=A_k M_k\cosh\lambda_k d+B_k M_k\sinh\lambda_k d+ \\ +\overline{A_k M_k\cosh\lambda_k d}+\overline{B_k M_k\sinh\lambda_k d}, \tag{5}$$

$$0=A_k\lambda_k M_k\sinh\lambda_k d+B_k\lambda_k M_k\cosh\lambda_k d+ \\ +\overline{A_k\lambda_k M_k\sinh\lambda_k d}+\overline{B_k\lambda_k M_k\cosh\lambda_k d}.$$

Solving (5), we find

$$A_k = \frac{\sigma_k^*}{M_k} \frac{\lambda_k \overline{\lambda_k}\left(\cosh \lambda_k d - \overline{\cosh \lambda_k d}\right)}{\Delta_k},$$
$$B_k = -\frac{\sigma_k^* \overline{\lambda_k}}{M_k} \frac{\left(\lambda_k \sinh \lambda_k d - \overline{\lambda_k \sinh \lambda_k d}\right)}{\Delta_k}, \tag{6}$$

where

$$\sigma_k^* = \sigma_k + \overline{\sigma_k}, \ \ \sigma_k = \int_{-1}^{1} \sigma(y) x_k(y) dy, \ \ M_k = L'(\lambda_k)/2\lambda_k,$$
$$\Delta_k = \lambda_k \overline{\lambda_k}\left(\cosh \lambda_k d - \overline{\cosh \lambda_k d}\right)^2 - \left(\lambda_k \sinh \lambda_k d - \overline{\lambda_k \sinh \lambda_k d}\right)\left(\overline{\lambda_k} \sinh \lambda_k d - \lambda_k \overline{\sinh \lambda_k d}\right).$$

It is obvious that Δ_k is real.

Substituting the coefficients A_k and B_k in formulae for displacements and stresses and isolating null-series [1], we obtain

$$U(x,y) = \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{\xi(\lambda_k, y)}{\lambda_k M_k} \frac{\mathrm{Re}\{\lambda_k S(\lambda_k, x)\}}{\Delta_k}\right\},$$
$$V(x,y) = \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{\chi(\lambda_k, y)}{M_k} \frac{\mathrm{Re}\,C(\lambda_k, x)}{\Delta_k}\right\},$$
$$\sigma_x(x,y) = \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{s_x(\lambda_k, y)}{M_k} \frac{\mathrm{Re}\,C(\lambda_k, x)}{\Delta_k}\right\}, \tag{7}$$
$$\sigma_y(x,y) = \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{s_y(\lambda_k, y)}{\lambda_k^2 M_k} \frac{\mathrm{Re}\{\lambda_k^2 C(\lambda_k, x)\}}{\Delta_k}\right\},$$
$$\tau_{xy}(x,y) = \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{t_{xy}(\lambda_k, y)}{\lambda_k M_k} \frac{\mathrm{Re}\{\lambda_k S(\lambda_k, x)\}}{\Delta_k}\right\},$$

where

$$C(\lambda_k, x) = \lambda_k \overline{\lambda_k}\left(\cosh \lambda_k d - \overline{\cosh \lambda_k d}\right)\cosh \lambda_k x - \overline{\lambda_k}\left(\lambda_k \sinh \lambda_k d - \overline{\lambda_k \sinh \lambda_k d}\right)\sinh \lambda_k x,$$
$$S(\lambda_k, x) = \lambda_k \overline{\lambda_k}\left(\cosh \lambda_k d - \overline{\cosh \lambda_k d}\right)\sinh \lambda_k x - \overline{\lambda_k}\left(\lambda_k \sinh \lambda_k d - \overline{\lambda_k \sinh \lambda_k d}\right)\cosh \lambda_k x.$$

Examples of Solving a Boundary Value Problem

Example 1. Even-Symmetric Deformation

Let the self-equilibrated normal load act on the right end of the rectangle (Fig. 1):

$$\sigma(y) = \begin{cases} y^4 - \dfrac{6\alpha^2}{5} y^2 + \dfrac{\alpha^4}{5} & (|y| \le \alpha), \\ 0 & (|y| \le \alpha). \end{cases} \tag{8}$$

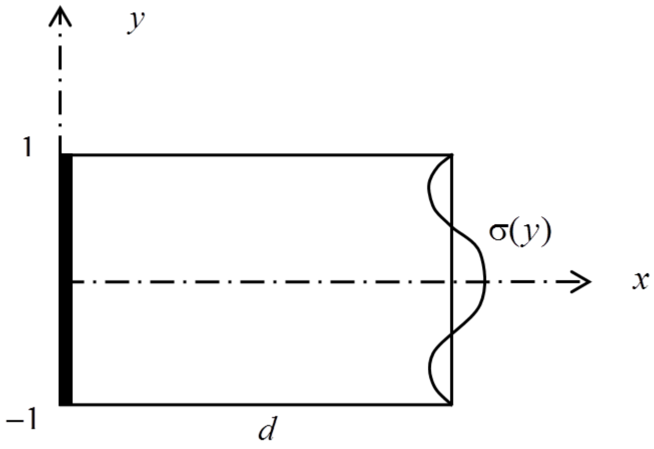

Figure 1. The scheme of the boundary value problem

The Papkovich–Fadle eigenfunctions $\xi(\lambda_k, y)$, $\chi(\lambda_k, y)$, $s_x(\lambda_k, y)$, $s_y(\lambda_k, y)$, $t_{xy}(\lambda_k, y)$ have the form

$$\begin{gathered}
\xi(\lambda_k, y) = \left(\frac{1-\nu}{2}\sin\lambda_k - \frac{1+\nu}{2}\lambda_k\cos\lambda_k\right)\cos\lambda_k y - \frac{1+\nu}{2}\lambda_k y\sin\lambda_k\sin\lambda_k y,\\
\chi(\lambda_k, y) = \left(\frac{1+\nu}{2}\lambda_k\cos\lambda_k + \sin\lambda_k\right)\sin\lambda_k y - \frac{1+\nu}{2}\lambda_k y\sin\lambda_k\cos\lambda_k y,\\
s_x(\lambda_k, y) = (1+\nu)\lambda_k\left\{(\sin\lambda_k - \lambda_k\cos\lambda_k)\cos\lambda_k y - \lambda_k y\sin\lambda_k\sin\lambda_k y\right\},\\
s_y(\lambda_k, y) = (1+\nu)\lambda_k\left\{(\sin\lambda_k + \lambda_k\cos\lambda_k)\cos\lambda_k y + \lambda_k y\sin\lambda_k\sin\lambda_k y\right\},\\
t_{xy}(\lambda_k, y) = (1+\nu)\lambda_k^2\left\{\cos\lambda_k\sin\lambda_k y - y\sin\lambda_k\cos\lambda_k y\right\},
\end{gathered} \tag{9}$$

where the numbers λ_k, $\overline{\lambda_k}$ $(\mathrm{Re}\,\lambda_k < 0)$ form the set $\left\{\pm\lambda_k; \pm\overline{\lambda_k}\right\}_{k=1}^{\infty} = \Lambda$ of all the complex zeros of the entire function $L(\lambda) = \lambda(\lambda + \sin\lambda\cos\lambda)$.

The functions biorthogonal to the Papkovich–Fadle eigenfunctions $\xi(\lambda_k, y)$, $\chi(\lambda_k, y)$, $s_x(\lambda_k, y)$, $s_y(\lambda_k, y)$ have the form [3]

$$\begin{gathered}
u_k(y) = \frac{1}{(1+\nu)}\left[\frac{\lambda_k\cos\lambda_k y}{\sin\lambda_k} - \left(\delta(y-1) + \delta(y+1)\right)\right],\\
v_k(y) = -\frac{\sin\lambda_k y}{(1+\nu)\sin\lambda_k},\ x_k(y) = \frac{\cos\lambda_k y}{2(1+\nu)\lambda_k\sin\lambda_k},\ t_k(y) = -\frac{\sin\lambda_k y}{2(1+\nu)\sin\lambda_k},
\end{gathered} \tag{10}$$

where δ is the Dirac delta function.

The numbers M_k and σ_k are equal to

$$M_k = \cos^2\lambda_k,\ \sigma_k = \frac{8\left((15 - 6\alpha^2\lambda_k^2)\sin\alpha\lambda_k + (\alpha^2\lambda_k^2 - 15)\alpha\lambda_k\cos\alpha\lambda_k\right)}{5(1+\nu)\lambda_k^6\sin\lambda_k}.$$

Substituting the found coefficients in formulae (7), we obtain the solution of the boundary value problem.

In Fig. 2 the distribution curves of the normal stresses $\sigma_x(1,y)$, $\sigma_x(0.9,y)$ and normal load $\sigma(y)$ are shown (it was assumed that $d=1$, $\alpha=0.5$, $\nu=\frac{1}{3}$).

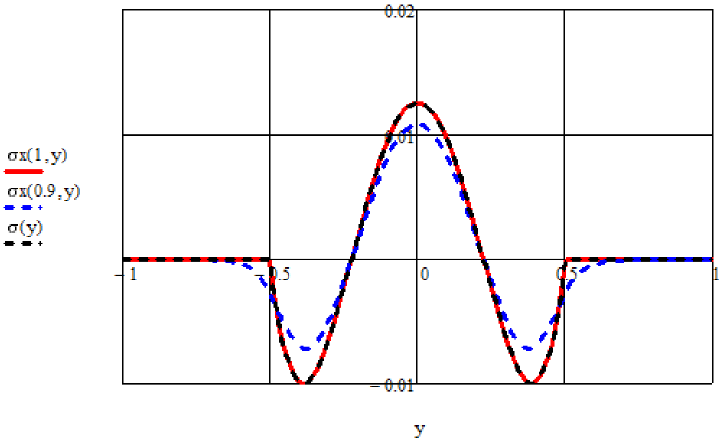

Figure 2. The distributions of the normal stresses $\sigma_x(1,y)$, $\sigma_x(0.9,y)$ and normal load $\sigma(y)$

Example 2. Odd-Symmetric deformation

Let the normal load that is self-equilibrated in moment act on the right end of the rectangle (Fig. 3):

$$\sigma(y)=y^5-\frac{10\alpha^3}{7}y^2+\frac{3}{7}y. \tag{11}$$

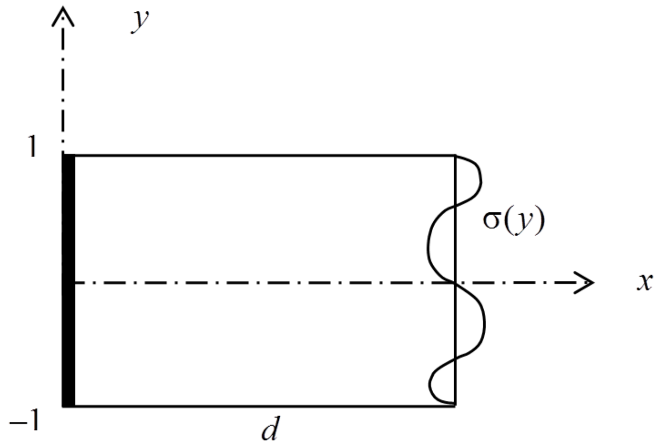

Figure 3. The scheme of the boundary value problem

In this case the Papkovich–Fadle eigenfunctions have the following form:

$$\xi(\lambda_k,y)=\left(\sin\lambda_k-\frac{1+\nu}{2}\lambda_k\cos\lambda_k\right)\sin\lambda_k y+\frac{1+\nu}{2}\lambda_k y\sin\lambda_k\cos\lambda_k y,$$

$$\chi(\lambda_k,y)=-\left(\frac{1-\nu}{2}\sin\lambda_k+\frac{1+\nu}{2}\lambda_k\cos\lambda_k\right)\cos\lambda_k y-\frac{1+\nu}{2}\lambda_k y\sin\lambda_k\sin\lambda_k y,$$

$$s_x(\lambda_k,y)=(1+\nu)\lambda_k\left\{(2\sin\lambda_k-\lambda_k\cos\lambda_k)\sin\lambda_k y+\lambda_k y\sin\lambda_k\cos\lambda_k y\right\}, \tag{12}$$

$$s_y(\lambda_k, y) = (1+\nu)\lambda_k^2\{\cos\lambda_k \sin\lambda_k y - y\sin\lambda_k \cos\lambda_k y\},$$
$$t_{xy}(\lambda_k, y) = (1+\nu)\lambda_k\{(\sin\lambda_k - \lambda_k\cos\lambda_k)\cos\lambda_k y - \lambda_k y \sin\lambda_k \sin\lambda_k y\},$$

and the numbers λ_k, $\overline{\lambda_k}$ $(\operatorname{Re}\lambda_k < 0)$ form the set $\{\pm\lambda_k; \pm\overline{\lambda_k}\}_{k=1}^{\infty} = \Lambda$ of all the complex zeros of the entire function $L(\lambda) = \lambda - \sin\lambda\cos\lambda$.

The functions biorthogonal to the Papkovich–Fadle eigenfunctions $\xi(\lambda_k, y)$, $\chi(\lambda_k, y)$, $s_x(\lambda_k, y)$, $s_y(\lambda_k, y)$ have the form [4]

$$u_k(y) = \frac{1}{(1+\nu)}\frac{\sin\lambda_k y}{\sin\lambda_k}, \; v_k(y) = \frac{1}{(1+\nu)}\frac{\cos\lambda_k y}{\lambda_k \sin\lambda_k},$$
$$x_k(y) = \frac{1}{2(1+\nu)\lambda_k^2}\left(\frac{\sin\lambda_k y}{\sin\lambda_k} - y\right), \; t_k(y) = \frac{1}{2(1+\nu)\lambda_k}\frac{\cos\lambda_k y}{\sin\lambda_k}. \qquad (13)$$

The numbers M_k and σ_k are equal to

$$M_k = \frac{\sin^2\lambda_k}{\lambda_k}, \; \sigma_k = \frac{8\left(\left(105 - 45\lambda_k^2 + \lambda_k^4\right)\sin\lambda_k + 5\left(2\lambda_k^2 - 21\right)\lambda_k\cos\lambda_k\right)}{7(1+\nu)\lambda_k^8 \sin\lambda_k}.$$

Substituting the found coefficients in formulae (7), we obtain the solution of the boundary value problem.

In Fig. 4 the distribution curves of the normal stresses $\sigma_x(1, y)$, $\sigma_x(0.9, y)$ and normal load $\sigma(y)$ are shown (it was assumed that $d = 1$, $\nu = \frac{1}{3}$).

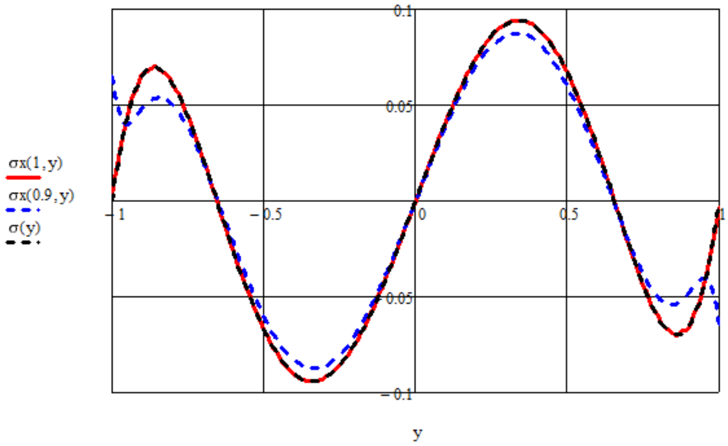

Figure 4. The distributions of the normal stresses $\sigma_x(1, y)$, $\sigma_x(0.9, y)$ and normal load $\sigma(y)$.

Conclusions

In both examples, on the end, the expanded functions $u = v = 0$, and we continue them by zero. Therefore, there will be no singularity at the angles if it is not introduced artificially by choosing a non-zero continuation outside the segment (end) [-1,1] in this or that way.

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