Finding the periodic solutions of delayed differential equations via solving optimization problem

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Abstract

This paper studies the computation of the periodic solutions of delay differential equations. By establishing the Poincaré map and considering the phase drift conditions we transform the computation of the periodic solution as an optimization problem with constraints. We propose a method to get approximately the initial function by function fitting. The results show that the proposed method improves the computational efficiency greatly compared to that using traditional function interpolation method.

Keywords: Time delay; optimization; periodic solution; initial functions; function fitting.

Introduction

The delay differential equation (DDE) is a complex infinite-dimensional system. Most systems with delays cannot be solved analytically, and need to be solved by numerical method. The periodic solution of the differential equation has extensive applications in many fields such as engineering practice, biology, economy and so on. The periodicity problem for DDEs is an infinite-dimensional problem because the delay differential equation system is defined in an infinite-dimensional space [1-7]. Currently, all known approaches for calculating the periodic solutions of the delay equations either use the Fourier series [8, 9] approximately locate the periodic solution or use a shooting approach discretizing an initial function on the interval. The first approach is quite efficient, but, it does not allow determining the stability of the periodic solutions. When applying the second approach, we can determine the stability of a periodic solution by computing Floquet multipliers. However, the shooting approach is expensive and inefficient when the periodic solution is not strongly attractive or unstable. In this paper, we transform the computation of the periodic solution of DDEs to the constraint optimization problem. We formulate a least squares function fitting method for determining the initial function. Compared with the traditional method, the computational complexity is reduced and the computational efficiency is improved.

Finding the Periodic Solutions of DDEs

We study the following system of DDEs with the time delay τ :

$$\begin{cases} \dot{x}(t) = F(t, x(t), x(t-\tau)) \\ x(\theta) = \varphi(\theta), \ -\tau \le \theta \le 0 \end{cases} \tag{1}$$

where x is an n-dimensional vector of variables; $\tau \ge 0$ is the time delay; φ is an initial function defined on $[-\tau, 0)$; $F(\cdot)$ is a n-dimensional vector-valued function. Let $x_t(\varphi)$ denote the solution segment defined by $x_t(\varphi) = x(t+\theta, \varphi)$, $-\tau \le \theta \le 0$, or in symbols,

$x_t(\varphi)(\theta) \in C([-\tau, 0], R^n)$, where C is the Banach space of continuous functions mapping the interval $[-\tau, 0] \to R^n$.

The computation of the periodic solution of DDES is to find the initial function $\varphi(\theta), (-\tau \le \theta \le 0)$ and period T such that $x_T(\varphi) = \varphi$, where $x_T(\varphi)$ is the solution segment defined on $[T - \tau, T]$ [10, 11].

In addition, a periodic solution of DDEs is also determined by the period T. The phase shift of any periodic solution is also a periodic solution. So in order to get accurate period, a phase condition $s(\varphi, T) = 0$ is needed to remove the indeterminacy.

Therefore, to find the periodic solution of (1), we use the following conditions

$$x_T(\varphi) - \varphi = 0, \quad s(\varphi, T) = 0. \tag{2}$$

This paper transforms the above conditions into the constraint optimization problem:

$$\min_{\varphi, T} \left\| x_T(\varphi) - \varphi \right\|^2, \ \text{s.t.} \ s(\varphi, T) = 0, \tag{3}$$

or

$$\min J(\varphi, T, \sigma) = \min \left[\left\| x_T(\varphi) - \varphi \right\|^2 + \sigma \left\| s(\varphi, T) \right\| \right], \tag{4}$$

where $\|\cdot\|$ is 2-norm, and σ is the penalty factor.

The aims of the above optimization problem is to find the initial function φ and the period T, under the constraint condition $s(\varphi, T) = 0$, get the minimum value of the function $\left\| x_T(\varphi) - \varphi \right\|^2$.

When the time delay is large, Eq. (2) may lose compactness because the phase space of Eq. (1) is infinite dimensional. We must consider this system in a finite dimensional space [12]. The first step of a numerical technique to compute a periodic solution is the discretization of the initial function φ. We choose grid spacing $h = \tau / (N - 1)$ and N mesh points $t_i = -\tau + (i - 1)h$. In the general shooting method, one usually uses φ_i ($i = 1, 2, \ldots, N$) of these mesh points to approach to the initial function φ by the interpolation method. But, this method is inefficient when the periodic solution is not strongly attractive or unstable. This is because in order to approach this periodic solution, we need a lot of φ_i (N is very large) to ensure the validity of the approximation. So, we must compute the $N \times N$ Jacobi matrix when the Newton method is used to solve problem (3), and the computational efficient is very low especially when N is large.

In this paper, we do not use the interpolation method, and think that the approximate values φ_i only need to reflect the motion trend of real initial function φ, and do not need the initial function accurately through φ_i in every approximation step. This paper constructs the initial function with the least squares function fitting, which the initial function obtained is the best approximation of φ_i in the sense of energy norm. The process is as follow (Fig. 1.).

Assuming that $\{\gamma_i\}_{i=1}^{m}$ is the independent basis vectors, we construct the initial function $\varphi = \sum_{j=1}^{m} \alpha_j \gamma_j$, and define

$$E(\hat{\alpha}) = (\varphi - \tilde{\varphi}, \varphi - \tilde{\varphi}) = \frac{1}{2} \sum_{i=1}^{N} \left[\varphi(t_i) - \tilde{\varphi}_i \right]^2 ,$$

where $\hat{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\tilde{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_N)$. $E(\hat{\alpha})$ can be further expressed as

$$\begin{aligned} E(\hat{\alpha}) &= (\varphi, \varphi) + (\tilde{\varphi}, \tilde{\varphi}) - 2(\varphi, \tilde{\varphi}) \\ &= \left(\sum_{p=1}^{m} \alpha_p \gamma_p , \sum_{r=1}^{m} \alpha_r \gamma_r \right) + (\tilde{\varphi}, \tilde{\varphi}) - 2\left(\sum_{p=1}^{m} \alpha_p \gamma_p , \tilde{\varphi} \right) \end{aligned} \quad (5)$$

Then

$$\begin{aligned} \frac{\partial E}{\partial \alpha_j} &= \left(\gamma_j , \sum_{r=1}^{m} \alpha_r \gamma_r \right) + \left(\sum_{p=1}^{m} \alpha_p \gamma_p , \gamma_j \right) - 2(\gamma_j, \tilde{\varphi}) \\ &= 2\left(\gamma_j , \sum_{r=1}^{m} \alpha_r \gamma_r \right) - 2(\gamma_j, \tilde{\varphi}) \quad (6) \\ &= 2 \sum_{r=1}^{m} (\gamma_j, \gamma_r) \alpha_r - 2(\gamma_j, \tilde{\varphi}) \end{aligned}$$

The minimum value of E is meet

$$\frac{\partial E}{\partial \alpha_j} = 0, \ j = 1, 2, \ldots, m . \quad (7)$$

so

$$\sum_{r=1}^{m} (\gamma_j, \gamma_r) \alpha_r - (\gamma_j, \tilde{\varphi}), \ j = 1, 2, \ldots, m . \quad (8)$$

Thus, for a linearly independent basis function $\{\gamma_i\}_{i=1}^{m}$, the unique solution $\{\alpha_r\}_{r=1}^{m}$ of the coefficients can be obtained by solving equation (8).

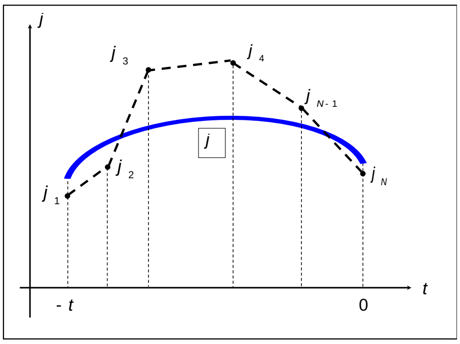

Figure.1 Construction of the initial function, the dotted line is the figure of classic (linear) interpolation method and solid lines is the figure by using fitting method

We use the Gram-Schmidt orthogonalization method to simplify the calculation, where the specific calculation details are ignored. The algorithm of commutating the periodic solution of the delay equation is as follows:

Step 0: Given the initial penalty factor $\sigma > 0$, amplification coefficient $\beta > 1$, permissible error $\varepsilon_1 > 0$, $\varepsilon_2 > 0$.

Step 1: Select a grid spacing $h = \tau / (N - 1)$, and define $t_i = -\tau + (i - 1)h$.

Step 2: Let the initial function value on the N grid points is $\tilde{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_N)^{\mathrm{T}}$, and period is T.

Step 3: Construct approximate initial function $\varphi(t)$ by least square method, and use an integral solver to calculate the approximate solution of Eq. (1) from 0 to T. Save the solution $x(t, \tilde{\varphi}, T)$ on $[T - \tau, T]$.

Step 4: Compute the approximate $x_i(t_i, \tilde{\varphi}, T)$ as an approximation of $x_T(\tilde{\varphi})$.

Step 5: Compute $J(\tilde{\varphi}, T) = \left\| x_T(\tilde{\varphi}) - \tilde{\varphi} \right\|^2 + \sigma \left\| s(\tilde{\varphi}, T) \right\|$. If $J(\tilde{\varphi}, T) < \varepsilon_1$, go to Step 6. Otherwise, if $J(\tilde{\varphi}, T) > \varepsilon$, use Newton Method to compute the search direction of the new iteration and update the initial function value and period. Return to Step 2.

Step 6: If $\sigma \left\| s(\tilde{\varphi}, T) \right\| < \varepsilon_2$, iteration is terminated, $P = (\tilde{\varphi}, T)^{\mathrm{T}}$ is the approximate optimal solution, and go to step 7. Otherwise, make $\sigma := \sigma\beta$ and return to Step 2.

Step 7: Save the initial value and the period, the iteration ends.

Numerical Results

An example is the group dynamics of the small world network, whose equation of motion as follows [13]

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \xi + V(t - \tau) - \mu\xi V^2(t - \tau). \tag{9}$$

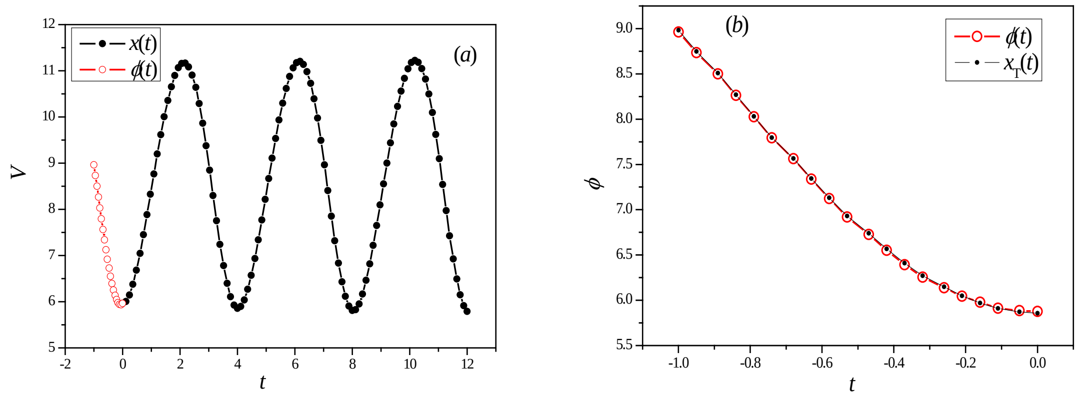

Figure. 2. Periodic solutions obtained by solving optimization problems.

According to the literature [13], when $\tau < \pi / 2$, the positive μ through the critical value $\mu^* = (\pi^2 - 4\tau^2)/(16\tau^2\xi^2)$, the system will admit a Hopf bifurcation and generate periodic solution. The period of periodic solution is 4τ. We can compute $\mu = 0.0408$, $T = 4$ by choosing $\xi = 3, \tau = 1$. By applying the optimization method proposed in this paper, we randomly select 11 initial function points in $[-\tau, 0]$, and use the third order polynomial as the fitting basis function to fit the initial function. The period is selected as 4, and thus the phase condition $s(\varphi, T) = 0$ is automatically satisfied. Figure 2 illustrates the periodic solution finding by using the proposed method. It is clear that the solution segment $x_T(\varphi)$ on $[T-\tau, T]$ is very close to the initial function φ, and the relative error is

$$\varepsilon \equiv \|x_T(\varphi)-\varphi\|/\|\varphi\| = 0.01351\%,$$

which indicates we find the approximate periodic solution.

For *N* original approximation of initial function, and m-order fitting basis functions, Table 1 gives the relative error. For the same fitting basis function m and different unknown number N, the error is almost the same; while for the same unknown number, the increasing of the numbers of the fitting basis function reduce obviously the error. However, for the third-order or fourth-order fitting basis function, the relative approximation of the calculation can reach about 0.01% for every *N*. Therefore, we can use low-order fitting basis function and relatively few unknowns to get better calculation accuracy.

Table 1. Relative error for different unknown variables and fitting orders

Error(%)	$N=11$	$N=15$	$N=21$	$N=31$
$m=3$	0.01351	0.01352	0.01351	0.01351
$m=4$	0.01163	0.01163	0.01148	0.01138
$m=5$	0.00841	0.00840	0.00836	0.00827
$m=6$	0.00540	0.00536	0.00535	0.00533

The eigenvalue of the operator $\left[\nabla^2 J(P)\right]$ is the Floquet multiplier of the periodic solution. According to the conclusion of [11], the linearized Poincaré operator of finite time-delay systems is compact. The Floquet multiplier is a point range centered around zero and $\mu=1$ is always a the Floquet multiplier of periodic solution. Based on this conclusion, we compute the first four Folquet multipliers of Eq. (9), as shown in Table 2. It can be seen that the results in Table 2 are consistent with the above conclusions, and that the largest Floqute multiplier is very close to 1, indicating the accuracy of the calculation.

Table 2. Comparisons of bifurcation points of Floquet multipliers (the first 4)

	$N=11$	$N=15$	$N=21$
$\lvert\mu_1\rvert$	0.98641	0.99768	0.99813
$\lvert\mu_2\rvert$	0.0510	0.02452	0.01283
$\lvert\mu_3\rvert$	0.00311	0.00120	0.00116
$\lvert\mu_4\rvert$	0.00085	0.00018	0.00013

Conclusions

In this paper, we propose a method to compute the periodic solutions of differential equations with delay. The delay differential equation is an infinite-dimensional system, so the problem of finding its periodic solution is infinite-dimensional problem. The periodic solution of the delay equation is different from the periodic solution of the ordinary differential equation, and the initial value of the system is defined on $[T-\tau, T]$. The computation of the periodic solution is to find the initial function and period T such that $x_T(\varphi)(\theta) = \varphi(\theta), -\tau \le \theta \le 0$. We consider the phase drift condition of the periodic solution, turn the problem of periodic solution into a constrained optimization problem, and give a specific algorithm to solve the problem of optimization. The traditional method uses function interpolation to approximate the initial function. When the periodic solution is not strongly attractive, the computational efficiency is very low. This paper uses the least squares fitting method to approximate the real initial function by a finite initial function value φ_i. The numerical experiments verify the validity of the proposed method. Using the function fitting method proposed in this paper, we can obtain the initial function of the periodic solution by a few unknowns and the appropriate order fitting function. When the solution of the periodic solution is strongly attracting, the low order fitting function is used; when the solution of the periodic solution is weak, or the periodic solution is unstable, the high order fitting function is used. The results show that the proposed method improves the computational efficiency greatly compared to that using traditional function interpolation method.

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Biography

Xu Xu received his Ph.D. from Jilin University, China, in 2003. He is currently a Professor at the College of Mathematics in Jilin University, China. His research interesting include: Multiscale Computation, Delayed Differential Equations; Complex Networks.