Plate/Shell Topology Optimization with Buckling and Frequency Constraints Based on Independent Continuous Mapping Method

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Abstract

In this paper, the topological optimization problem considering the stability and vibration characteristics of structure is studied. Firstly, the topology optimization model with the minimum structure volume as object is established based on independent, continuous and mapping (ICM) method, which subjects to the critical buckling load and frequencies as constraints. The filter functions with composite exponential function of elemental mass matrix, elemental stiffness matrix and elemental geometric matrix would be introduced, by which the three matrixes are updated in iteration putted into the topology optimization of differential equation to analyse the design sensitivity and optimize the structure. Secondly, the optimal model is conversed into the quadratic programming with the introducing of filter functions, Taylor expansion and sensitivity analysis. And then, the mathematical model is solved by dual sequence quadratic programming (DSQP) algorithm. Finally, the bisection method is applied to reduce searching region of threshold space to find the optimal mapping from "continuous" to "discrete". Numerical examples are given to illustrate the feasibility and efficiency of the proposed method.

Keywords: Topology optimization, Critical buckling load, Frequency constraints, ICM method, Plate/shell structure

1 Introduction

The topology optimization of continuum structure can reduce the cost of structure with little waste of material, owing to the ability of finding an optimum path transferring load on the base structure with given constraints [1][2]. Topology optimization has widely prospect in the application of automobile, machinery, aerospace, civil engineering and etc. The representative topology optimization methods for continuous structure include homogenization method [3], variable density method (including SIMP and RAMP interpolation model) [4]-[5], evolutionary structural optimization (ESO) [6], level set method [7], moving morphable components(MMC) [8], phase field method [9] etc.The plate/shell structures become more and more popular among the engineering designers for simple structure form, and the stability and dynamics characteristics of structure are considered as the significant factors for assessing the plate/shell structures design [10]-[14]. Therefore, considering buckling and frequency constraints is important for the plate/shell topology optimization.

In this paper, the plate/shell topology optimization model, which takes the critical buckling load and natural frequencies as constraints, is established based on independent, continuous and mapping (ICM) method. The discrete topology optimization model is translated into continuous model with the introduction of composite exponential filter function. The sensitivity analysis and first-order Taylor expansion are applied to explicit the buckling and frequency constraints, and the optimal model is solved by conversing into the quadratic programming model. Bisection inversion strategy is used to realize the inversion of topology variables from “continuous” to “discrete”.

2 Mathematical model of multi-constraints topology optimization

2.1 ICM method and CEF filter function

The filter functions are the key technology of ICM, which realize the mapping of topological variables from “discrete” to “continuous” and establish the relationship between the optimal model and the topological variables and the physical properties of the element. Here, the composite exponential function (CEF) is selected as filter functions to realize the approximate transformation of the topological variables from 0/1 to (0, 1].

$$f(t_i) = (e^{t_i/\alpha} - 1)/(e^{1/\alpha} - 1)\ , \tag{2}$$

where α is the given positive constant.

The element volume, stiffness matrix, geometric stiffness matrix, and mass matrix v_i, $\boldsymbol{k}_i$, $\boldsymbol{g}_i$, $\boldsymbol{m}_i$ of i-th element in the optimal process are recognized by the filter functions as follows

$$v_i = f_v(t_i)v_i^0,\ \boldsymbol{k}_i = f_k(t_i)\boldsymbol{k}_i^0,\ \boldsymbol{g}_i = f_g(t_i)\boldsymbol{g}_i^0,\ \boldsymbol{m}_i = f_m(t_i)\boldsymbol{m}_i^0\ , \tag{3}$$

where v_i^0, $\boldsymbol{k}_i^0$, $\boldsymbol{g}_i^0$, $\boldsymbol{m}_i^0$ represent the initial element volume, stiffness matrix, geometric stiffness matrix, mass matrix of i-th element, respectively, and $f_v(t_i)$, $f_k(t_i)$, $f_g(t_i)$, $f_m(t_i)$ are the corresponding filter functions.

2.2 Establishment of the topological optimization model

In order to guarantee the optimal structure meeting the requirements of stability and dynamic characteristics, the topological optimization model with the minimum structure volume as object , which subjects to the buckling load and frequencies constraints, is established.

$$\begin{cases} \text{find} \quad \boldsymbol{t} \in E^N\ , \\ \text{make} \quad V \to \min\ , \\ \text{s.t.} \quad P_{\text{cr}1} \geq \underline{P}_1\ , \\ \qquad\ \ \gamma_l \geq \underline{\gamma}_l \quad (l = 1, \cdots, L)\ , \\ \qquad\ \ t_{\min} \leq t_i \leq 1 \quad (i = 1, \cdots, N)\ , \end{cases} \tag{1}$$

where $\boldsymbol{t}$ is the vector of topological variables and E^N denotes N-dimensional Euclidean space, V is the total volume of the structure, $P_{\text{cr}1}$ and $\underline{P}_1$, respectively, present the 1-th critical buckling load and lower limit of buckling critical load, γ_l and $\underline{\gamma}_l$ are the l-th natural frequency and lower limit of l-th natural frequency, respectively, L is the total number of the frequency constraints and N is the total number of elements.

By taking advantage of the relations between the filter functions and physical properties of the element, the optimal model (1) can be rewritten as

$$\begin{cases} \text{find} \quad \boldsymbol{t} \in E^N, \\ \text{make} \;\; V = \sum_{i=1}^{N} f_v\left(t_i\right) v_i^0 \to \min , \\ \text{s.t.} \;\; P_{\text{cr1}}\left(f_k\left(t_i\right), f_g\left(t_i\right)\right) \geq \underline{P}_1 , \\ \qquad \gamma_l(f_k\left(t_i\right), f_m\left(t_i\right)) \geq \underline{\gamma}_l \quad (l = 1, \cdots, L) , \\ \qquad t_{\min} \leq t_i \leq 1 \quad \left(i = 1, \cdots, N\right) . \end{cases} \tag{4}$$

In the process of optimization, the reciprocal variable of the stiffness filter function $x_i = 1/ f_k(t_i)$ is introduced as design variable.

3 Standardization and solution of the optimization model

3.1 Explicit approximation of buckling constraints

The finite element method is applied for the mechanical analysis of plate/shell structure. The buckling characteristic equation for the linear elastic plate/shell structure is expressed as

$$(\boldsymbol{K} + \lambda_1 \boldsymbol{G})\boldsymbol{u}_1 = \boldsymbol{0} , \tag{5}$$

where $\boldsymbol{K}$ and $\boldsymbol{G}$ denote the structural stiffness matrix and geometric stiffness matrix respectively, λ_1 is the 1-th buckling critical load factor and $\boldsymbol{u}_1$ represents the corresponding eigenvector of λ_1. In the linear buckling finite element analysis, the λ_1 is an important index to evaluate the structural buckling performance, which is linearly related to P_{cr1}.

$$P_{\text{cr1}} = \lambda_1 \times P, \;\; \underline{P}_1 = \underline{\lambda}_1 \times P, \tag{6}$$

where P is the given external mechanical load and $\underline{\lambda}_1$ is the lower limit of 1-th buckling critical load factor. Therefore, the critical buckling constraint can be simplified as the constraint of buckling critical load factor.

$$\lambda_1 \geq \underline{\lambda}_1. \tag{7}$$

Equation (5) shows that the buckling critical load factor is associated to the structural stiffness matrix and geometric stiffness matrix. Then the sensitivity analysis and first-order Taylor expansion are used to get the explicit equations of the buckling constraint.

$$\lambda_1(\boldsymbol{x}) \approx \lambda_1(\boldsymbol{x}^{(v)}) + \sum_{i=1}^{N} A_{i1}^{(v)} \frac{1}{x_i^{(v)}} x_i - \sum_{i=1}^{N} A_{i1}^{(v)} , \tag{8}$$

where $A_{i1} = \dfrac{U_{i1} + \lambda_1 \beta(x_i) V_{i1}}{V_{\sum 1}}$, $\beta(x_i) = \dfrac{f_g^{'}(t_i) f_k(t_i)}{f_g(t_i) f_k^{'}(t_i)} = \dfrac{\gamma_k (e^{t_i/\gamma_k} - 1)}{\gamma_g (e^{t_i/\gamma_g} - 1)} e^{t_i(\frac{1}{\gamma_g} - \frac{1}{\gamma_k})}$. $U_{i1} = 0.5\boldsymbol{u}_1^{\mathrm{T}} \boldsymbol{k}_i \boldsymbol{u}_1$ and $V_{i1} = 0.5\boldsymbol{u}_1^{\mathrm{T}} \boldsymbol{g}_i \boldsymbol{u}_1$ are the 1-th mode strain energy and geometric strain energy for i-th element, respectively, $V_{\sum 1} = \boldsymbol{u}_1^{\mathrm{T}} \boldsymbol{G} \boldsymbol{u}_1$ is the total geometric strain energy of structure for the 1-th buckling mode, which can be obtain from the results of finite element buckling analysis. Then the buckling constraints in optimal model (4) can be written as

$$\sum_{i=1}^{N} c_{i1} x_i \geq d_1 \ , \tag{9}$$

where $c_{i1} = A_{i1}^{(v)} \dfrac{1}{x_i^{(\upsilon)}}$, $d_1 = \underline{\lambda}_1 - \lambda_1(\boldsymbol{x}^{(\upsilon)}) + \sum_{i=1}^{N} A_{i1}^{(v)}$.

3.2 Explicit approximation of frequencies constraints

For linear multi-degree of freedom system without considering damping and additional dynamic load, the frequency equation of structure can be expressed as follows

$$(\boldsymbol{K} - \xi_l \boldsymbol{M})\boldsymbol{p}_l = \boldsymbol{0} \ , \tag{10}$$

where $\boldsymbol{M}$ is the mass matrix of structure. ξ_l is the l-th frequency eigenvalue and $\boldsymbol{p}_l$ is the eigenvector corresponding ξ_l. Considering the relationship between natural frequency γ_l and frequency eigenvalue ξ_l meets $\xi_l = (2\pi\gamma_l)^2$, and $\gamma_l > 0$, $\underline{\gamma}_l > 0$, the frequency constraints equation can be transformed into

$$\xi_l(f_k(t_i), f_m(t_i)) \geq \underline{\xi}_l \quad (l = 1, \cdots, L) \ , \tag{11}$$

where $\underline{\xi}_l = (2\pi\underline{\gamma}_l)^2$ is the corresponding lower limit of frequency eigenvalue constraint.

The explicit method of critical buckling load factor is applied to get the explicit function of frequency eigenvalue. When the eigenvector is satisfied $\boldsymbol{p}_l^{\mathrm{T}} \boldsymbol{M} \boldsymbol{p}_l = 1$, the frequency eigenvalue can be expressed as follows

$$\xi_l(\boldsymbol{x}) \approx \xi_l(\boldsymbol{x}^{(v)}) + \sum_{i=1}^{N} B_{il}^{(v)} \frac{1}{x_i^{(v)}} x_i - \sum_{i=1}^{N} B_{il}^{(v)} \ (l = 1, \cdots, L) \ . \tag{12}$$

where $B_{il} = \dfrac{2}{x_i^{(v)}}(\phi(x_i^{(v)})D_{il} - U_{il})$, $\phi(x_i) = \dfrac{f_m{}'(t_i) f_k(t_i)}{f_m(t_i) f_k{}'(t_i)} = \dfrac{\gamma_k(e^{t_i/\gamma_k} - 1)}{\gamma_m(e^{t_i/\gamma_m} - 1)} e^{t_i(\frac{1}{\gamma_m} - \frac{1}{\gamma_k})}$. $U_{il} = \dfrac{1}{2} \boldsymbol{p}_l^{\mathrm{T}} \boldsymbol{k}_i \boldsymbol{p}_l$ and $D_{il} = \dfrac{1}{2} \boldsymbol{p}_l^{\mathrm{T}} \boldsymbol{m}_i \boldsymbol{p}_l$ represent the strain energy and the kinetic energy of i-th element corresponding to the j-th frequency, respectively.

Let $B_{il} = \dfrac{2}{x_i^{(v)}}(\phi(x_i^{(v)})D_{il} - U_{il})$, $g_l = \underline{\xi}_l - \xi_l(\boldsymbol{x}^{(v)}) - \sum_{i=1}^{N} 2(\phi(x_i^{(v)})D_{il} - U_{il})$, and the frequency constraints can simplified as follows

$$\sum_{i=1}^{N} B_{il} x_i \geq g_l \ , \tag{13}$$

3.3 The standardization of the optimal model

The second-order Taylor expansion is applied to obtain the approximate explicit function of structure volume. Then the model (4) can be translated into the standard quadratic programming model.

$$\begin{cases} \text{find} \quad \boldsymbol{x} \in E^N \ , \\ \text{make} \quad V = \sum_{i=1}^{N} \left(a_i x_i^2 + b_i x_i \right) \to \min \ , \\ \text{s.t.} \quad \sum_{i=1}^{N} c_{i1} x_i \ge d_1 \quad , \\ \qquad \sum_{i=1}^{N} B_{il} x_i \ge g_l \quad \left(l = 1, \ldots, L \right) \ , \\ \qquad 1 \le x_i \le \overline{x}_i \quad \left(i = 1, \ldots, N \right) \ . \end{cases} \tag{14}$$

where

$$\begin{aligned} a_i &= \frac{1}{2} \frac{\gamma_k}{\gamma_m} \frac{m_i^0}{(x_i^{(v)})^4} \frac{e^{1/\lambda_k - 1}}{e^{1/\lambda_m - 1}} \left(\frac{e^{1/\gamma_k} - 1}{x_i^{(v)}} + 1 \right)^{\frac{\gamma_k}{\gamma_m} - 2} \times \left[\left(\frac{\lambda_k}{\lambda_m} + 1 \right) (e^{1/\gamma_k} - 1) + 2 x_i^{(v)} \right] , \\ b_i &= \frac{1}{2} \frac{\lambda_k}{\lambda_m} \frac{m_i^0}{(x_i^{(v)})^3} \frac{e^{1/\gamma_k - 1}}{e^{1/\gamma_m - 1}} \left(\frac{e^{1/\gamma_k} - 1}{x_i^{(v)}} + 1 \right) \times \left[\left(\frac{\gamma_k}{\gamma_m} + 1 \right) (e^{1/\gamma_k} - 1) + 3 x_i^{(v)} \right] . \end{aligned} \tag{15}$$

For the above mathematical model established, the number of design variables is much bigger than that of constraints. In order to reduce the number of design variables and improve the computational efficiency, the duality theory is introduced to convert the Eq.(15) into dual optimization model and dual sequence quadratic programming (DSQP) algorithm is applied to solve this optimal model.

3.4 Bisection inversion strategy

The optimal results obtained from Eq.(15) have some "continuous" topological variables, which affect the mechanical properties of the optimal structure to some degree. Here the bisection inversion strategy is applied to bisect threshold space to get a suitable filter threshold value to realize the mapping from "continuous" to "discrete", with the purpose of obtaining the optimal discrete structure.

The function $\mathrm{g}(T)$ is defined to indicate the minimum difference between values of constraints and those of discrete structure's mechanical properties when given the filter threshold value T. Here the concrete format of equation of $\mathrm{g}(T)$ can be written as follows

$$g(T) = \min[(P_{\mathrm{cr1}}(T) - \underline{P}_1) / \underline{P}_1 \times 100\% \ , \ (\gamma_l(T) - \underline{\gamma}_l) / \underline{\gamma}_l \times 100\% \ \ (l = 1, \cdots, L) \ ] \tag{16}$$

The flowchart of the bisection inversion strategy is illustrated as Figure 1 and the convergence condition of $\mathrm{g}(T)$ is given as

$$0 < g(T) <= \varepsilon \tag{17}$$

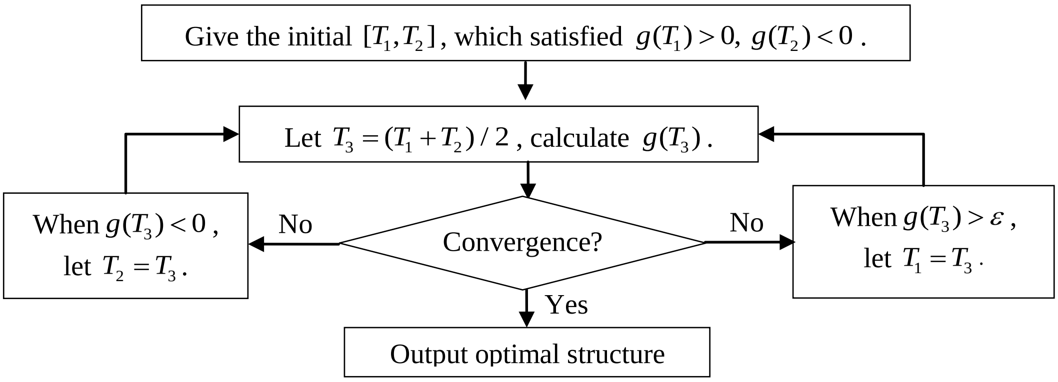

Figure 1. Flowchart of the bisection inversion strategy

4 Numerical example

The design region is a plane elastic body with size $20 \times 40 \times 1\text{mm}^3$ as shown in Figure 2. The Young's modulus $E = 68890\text{Mpa}$, Poisson's ratio μ=0.3, and the density ρ=1000kg/m^3. A concentrated mass M=2E-3kg is attached in the midpoint of the top boundary. The bottom boundary is fixed and the concentrated force 1000N is applied at the same position of concentrated mass. The basic structure is divided into 60×120 CQUAD4 elements. After finite element analysis, the first-order buckling load of the basic structures is 180.76N and The first three order frequencies of the basic structure are $\gamma_1 = 236.6\text{Hz},\ \gamma_2 = 2213.8\text{Hz},\ \gamma_3 = 2332.5\text{Hz}$. $\underline{P}_1 = 130\text{N}$ is defined as buckling constraint and $\underline{\gamma}_l = 0.9 \times \gamma_l\ \ (l = 1, 2, 3)$ is defined as frequency constraint.

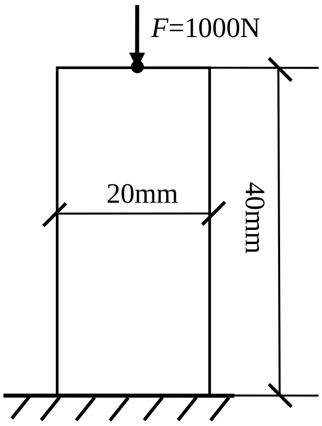

Figure 2. Design region

Table 1 gives the optimal results with different kinds of multi-constraints. The intermediate results and optimal topology configuration are illustrated in Figure 3. Figures 4 and 5 show the iteration history curves of volume and buckling load.

Comparing the optimal results in table 1, it is obvious found that the all the optimal structures satisfy the buckling and frequency constraints when different order of frequency constraint is given. And Figures 4 and 5 indicate that the processes of iteration are stable convergence. Those indicate that the optimization method is effective for solving the multi-constraints problem. From Figure 3, it can be found that the optimal structures with 2-th frequency and 3-th frequency constraints are similar, but all are different from that with 1-th frequency constraint. The main reason for this phenomenon is that the magnitudes of 2-th frequency and

3-th frequency of basic structure are approximately equivalent, meanwhile, the corresponding frequency constraint values are also almost same.

Table 1. Optimal topology process with different kinds of multi-constraints a Buckling & 1-th frequency b Buckling & 2-th frequency c Buckling & 3-th frequency

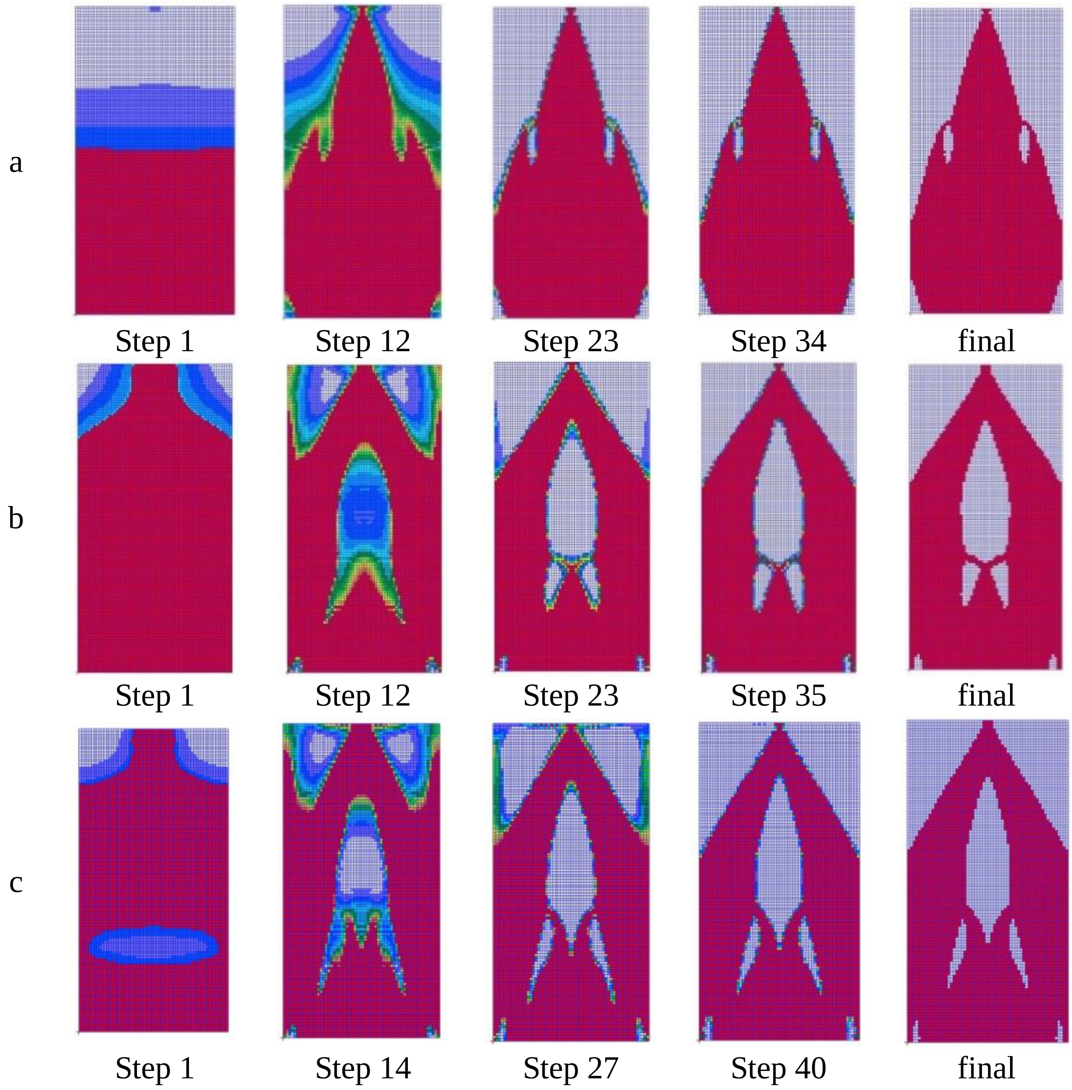

Table 2. Results of topology optimization with different kinds of multi-constraints a Buckling & 1-th frequency b Buckling & 2-th frequency c Buckling & 3-th frequency

Constraint type	a	b	c
Iteration number	34	35	40
Volume/mm^3	515.5	531.5	529.2
Buckling load/N	130.8	130.7	130.8
1-th frequency /Hz	229.3	223.2	223.2
2-th frequency /Hz	1643.2	2169.9	2085.1
3-th frequency /Hz	1647.4	2195.2	2174.3

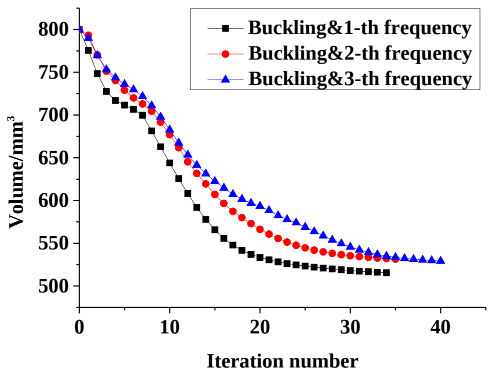

Figure 4. Iterative history curves of volume with different kinds of multi-constraints

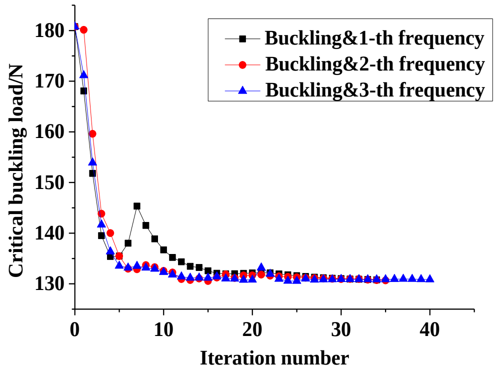

Figure 5. Iterative history curves of buckling load different kinds of multi-constraints

5 Conclusions

The plate/shell topology optimization model with minimum structural volume as objective, subjected to the critical buckling load and natural frequencies, is investigated in this paper. The composite exponential filter functions, Taylor expansion and sensitivity analysis are used to make the optimal model explicit and establish the standard quadratic programming model. And bisection inversion strategy is applied to realize the intelligent mapping of topological variables. The results of example demonstrate that proposed method for plate/shell topology optimization with buckling and frequencies constraints is feasible and valid.

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