Identification of Voids Using Topological Derivative and Level Set Method

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Abstract

An approach has been developed to identify the position of voids in structures using topological derivative and level set method. The position of voids is identified by solving an optimization problem. The level set method is applied in the present approach to represent the voids. The topology derivative of the objective function is used to produce voids in the problem domain. The shape derivative of the objective function is used to evolve the boundary of the voids. This approach has been applied to the voids identification of two-dimensional (2D) structures, the examples with multiple voids are considered. The results indicate that the voids in structure can be identified effectively by the present algorithm.

Keywords: void identification; level set method; topological derivative; shape derivative; velocity field

1. Introduction

Identification of voids in structures has been studied by many researchers [1, 2]. Among these identification schemes the solving of forward problems commonly used several parameters to explicitly represent the shape and geometry of the voids in structures. If the boundaries of the voids are curves or surfaces rather than straight lines, it would need too many parameters to describe. So this explicit expression scheme is inefficient for multiple voids identification. This disadvantages can be overcame by Level set method [3, 4] that was proposed firstly by Osher and Sethian. In the past three decades, level set method has been applied widely to structure topological optimization [5, 6] and numerical simulation. It also has been applied to several inverse problems [7, 8].

The limitation of the level set method based on shape derivative is that there is no change of topology in the problem domain. The voids can only merge and disappear but cannot split when the boundary of the voids evolves. The results may fall into a local minimum which corresponding to the initial topology. The topological derivative [9, 10] has been used to solve this problem in present paper. The main idea is change the topology of the domain by adding a small hole at the points where the topological derivatives are most negative.

The purpose of this paper is to develop an algorithm to identify the position of voids in structures using the level set method with topological derivative. In the present algorithm, the level set function is used to implicitly represent the location of voids and the FEM is used to solving the forward problem. The identification of voids can be transformed into a minimum optimization problem, in which the least square errors of displacement field of problem are taken as the objective function. Then, the identification problem is solved by minimizing the objective function. Displacement and adjoint displacement fields are obtained by solving the forward problem using FEM. A velocity field that evolves the level set function is derived by analyzing the shape derivative of the objective function. The topological derivative of the objective function has been introduced to produce voids every several iterations. Finally the location of voids can be obtained by evolving the level set function iteratively. A numerical example about voids identification of 2D problem is given to verify the present algorithm.

The outline of this paper is as follows. Section 2 states the forward problem and the shape derivative. Section 3 introduces the topological derivative. Section 4 describes the algorithm based on the level set method. Section 5 gives the 2D numerical example. Section 6 is the conclusion.

2. Problem statement and shape derivative

2.1 Problem statement

The model of forward problem employed in this paper is the linear elasticity problem, which can be described by the following state equation

$$\begin{cases} -\mathrm{div}(\mathrm{A}\,\mathbf{e}(\mathbf{u}))=\mathbf{g} & \text{in } \Omega \\ \mathrm{A}\,\mathbf{e}(\mathbf{u})\mathbf{n}=\mathbf{f} & \text{on } \Gamma_{\mathrm{N}} \\ \mathrm{A}\,\mathbf{e}(\mathbf{u})\mathbf{n}=0 & \text{on } \Gamma_{0} \\ \mathbf{u}=0 & \text{on } \Gamma_{\mathrm{D}} \end{cases} \tag{1}$$

where $\mathbf{u}$ is the displacement field, $\mathbf{e}(\mathbf{u})$ is the strain tensor, $\mathbf{A}$ is the elasticity tensor, $\mathbf{g}$ is the body force, $\mathbf{f}$ is the boundary traction force, $\mathbf{n}$ is the outward unit normal vector of the boundary, Ω is the solid domain with a linear isotropic material, the boundary of Ω is made of three parts

$$\partial\Omega=\Gamma_{\mathrm{N}}\cup\Gamma_{0}\cup\Gamma_{\mathrm{D}} \tag{2}$$

in which Γ_{N} is the force boundary, Γ_{0} is the free boundary and Γ_{D} is the displacement boundary.

2.2 Shape derivative

The identification problem can be transformed into a minimum optimization problem. The objective function of this optimization problem is denoted by $L(\Omega)$. It is defined as the least square error form with augmented Lagrange terms.

$$L(\Omega)=\sum_{i=1}^{m}(\mathbf{u}_i-\mathbf{u}_{0i})^2+\int_{\Omega}\mathbf{p}\cdot(-\mathrm{div}(\mathbf{A}\,\mathrm{e}(\mathbf{u}))-\mathbf{g})\,\mathrm{dx}+\int_{\Gamma_N}\mathbf{p}\cdot(\mathbf{A}\,\mathrm{e}(\mathbf{u})\mathbf{n}-\mathbf{f})\mathrm{ds}$$
$$+\int_{\Gamma_0}\mathbf{p}\cdot\mathbf{A}\,\mathrm{e}(\mathbf{u})\mathbf{n}\mathrm{ds}+\int_{\Gamma_D}\boldsymbol{\lambda}\cdot\mathbf{u}\,\mathrm{ds} \tag{3}$$

where $\mathbf{u}_{0i}$ is the measured displacement field, $\mathbf{p}$ and $\boldsymbol{\lambda}$ is the Lagrange multiplier and is also the adjoint displacement field.

Integrating the third term by part, we get

$$L(\Omega)=\sum_{i=1}^{m}(\mathbf{u}_i-\mathbf{u}_{0i})^2+\int_{\Omega}\mathbf{A}\,\mathrm{e}(\mathbf{u})\cdot\mathrm{e}(\mathbf{p})\,\mathrm{dx}-\int_{\Omega}\mathbf{p}\cdot\mathbf{g}\,\mathrm{dx}-\int_{\Gamma_N}\mathbf{p}\cdot\mathbf{f}\mathrm{ds}$$
$$+\int_{\Gamma_D}(\boldsymbol{\lambda}\cdot\mathbf{u}-\mathbf{p}\cdot\mathbf{A}\,\mathrm{e}(\mathbf{u})\mathbf{n})\mathrm{ds} \tag{4}$$

Integrating the third term by part again, we get

$$L(\Omega)=\sum_{i=1}^{m}(\mathbf{u}_i-\mathbf{u}_{0i})^2-\int_{\Omega}\mathbf{u}\cdot\mathrm{div}(\mathbf{A}\,\mathrm{e}(\mathbf{p}))\,\mathrm{dx}-\int_{\Omega}\mathbf{p}\cdot\mathbf{g}\,\mathrm{dx}+\int_{\Gamma_N}(\mathbf{u}\cdot\mathbf{A}\,\mathrm{e}(\mathbf{p})\mathbf{n}-\mathbf{p}\cdot\mathbf{f})\,\mathrm{ds}$$
$$+\int_{\Gamma_0}\mathbf{u}\cdot\mathbf{A}\,\mathrm{e}(\mathbf{p})\mathbf{n}\,\mathrm{ds}+\int_{\Gamma_D}(\boldsymbol{\lambda}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{A}\,\mathrm{e}(\mathbf{p})\mathbf{n}-\mathbf{p}\cdot\mathbf{A}\,\mathrm{e}(\mathbf{u})\mathbf{n})\,\mathrm{ds} \tag{5}$$

The partial derivative of L, using the form given by Eq. (5), with respect to $\mathbf{u}$ in the direction $\boldsymbol{\theta}$ is

$$\frac{\partial L}{\partial\mathbf{u}}(\boldsymbol{\theta})=\sum_{i=1}^{m}2\cdot(\mathbf{u}_i-\mathbf{u}_{0i})\cdot\boldsymbol{\theta}-\int_{\Omega}\mathrm{div}(\mathbf{A}\,\mathrm{e}(\mathbf{p}))\cdot\boldsymbol{\theta}\,\mathrm{dx}+\int_{\Gamma_N}\mathbf{A}\,\mathrm{e}(\mathbf{p})\mathbf{n}\cdot\boldsymbol{\theta}\,\mathrm{ds}$$
$$+\int_{\Gamma_0}\mathbf{A}\,\mathrm{e}(\mathbf{p})\mathbf{n}\cdot\boldsymbol{\theta}\,\mathrm{ds}+\int_{\Gamma_D}((\boldsymbol{\lambda}+\mathbf{A}\,\mathrm{e}(\mathbf{p})\mathbf{n})\cdot\boldsymbol{\theta}-\mathbf{p}\cdot\mathbf{A}\,\mathrm{e}(\boldsymbol{\theta})\mathbf{n})\,\mathrm{ds} \tag{6}$$

Letting Eq.(6) equal to zero, we get the adjoint equation for the adjoint field $\mathbf{p}$.

$$\begin{cases}-\mathrm{div}(\mathrm{A}\,\mathbf{e}(\mathbf{p}))=0 & \text{in }\Omega\\ \mathrm{A}\,\mathbf{e}(\mathbf{p})\mathbf{n}=-2\cdot(\mathbf{u}_i-\mathbf{u}_{0i}) & \text{on }\mathrm{N}_i\\ \mathrm{A}\,\mathbf{e}(\mathbf{p})\mathbf{n}=0 & \text{on }\Gamma_0\cup\Gamma_N\setminus\mathrm{N}_i\\ \mathbf{p}=0 & \text{on }\Gamma_D\end{cases} \tag{7}$$

We also get the adjoint equation for the adjoint field $\boldsymbol{\lambda}$.

$$\boldsymbol{\lambda} + \mathbf{A}\,\mathrm{e}(\mathbf{p})\mathbf{n} = 0 \quad \text{on}\,\Gamma_{\mathrm{D}} \tag{8}$$

The partial derivative of L, using the form given by Eq. (4), with respect to Ω in the direction $\mathbf{V}$ is

$$\begin{aligned}\frac{\partial L}{\partial \Omega}(\mathbf{V}) = \int_{\partial\Omega} (\mathbf{A}\,\mathrm{e}(\mathbf{u})\cdot \mathrm{e}(\mathbf{p}) - \mathbf{p}\cdot\mathbf{g})\mathbf{V}\cdot\mathbf{n}\mathrm{ds} - \int_{\Gamma_{\mathrm{N}}} (\frac{\partial(\mathbf{p}\cdot\mathbf{f})}{\partial\mathbf{n}} + \kappa\mathbf{p}\cdot\mathbf{f})\mathbf{V}\cdot\mathbf{n}\mathrm{ds} \\ - \int_{\Gamma_{\mathrm{D}}} (\frac{\partial h}{\partial \mathbf{n}} + \kappa h)\mathbf{V}\cdot\mathbf{n}\mathrm{ds}\end{aligned} \tag{9}$$

where κ is the curvature of boundary, and $h = \mathbf{u}\cdot\mathbf{A}\,\mathrm{e}(\mathbf{p})\mathbf{n} + \mathbf{p}\cdot\mathbf{A}\,\mathrm{e}(\mathbf{u})\mathbf{n}$. Since only the free boundary Γ_0 is the identified boundary, Eq. (9) becomes

$$\frac{\partial L}{\partial \Omega}(\mathbf{V}) = \int_{\Gamma_0} (\mathbf{A}\,\mathrm{e}(\mathbf{u})\cdot \mathrm{e}(\mathbf{p}) - \mathbf{p}\cdot\mathbf{g})\mathbf{V}\cdot\mathbf{n}\mathrm{ds} \tag{10}$$

3. Topological derivative

We give a brief review of this method that we shall call topological gradient method. Consider an open set $\Omega \subset \mathbf{R}^2$ and a point $x_0 \in \Omega$. Introduce a fixed hole $w \subset \mathbf{R}^2$, a smooth open bounded subset containing the origin. For $\rho > 0$ we define the translated and rescaled hole

$$w_\rho = x_0 + \rho w \tag{11}$$

Then we define the perforated domain

$$\Omega_\rho = \Omega \setminus w_\rho \tag{12}$$

In the framework of structural optimization we put Neumann boundary conditions on ∂w_ρ. The objective function $L(\Omega_\rho)$ is computed with the elastic displacement u_ρ, solution of the following elasticity problem

$$\begin{cases} -div(\mathrm{A}\,\mathbf{e}(\mathbf{u}_\rho)) = \mathbf{f} & \text{in}\,\Omega_\rho \\ \mathbf{u}_\rho = 0 & \text{on}\;\Gamma_D \\ \mathrm{A}\,\mathbf{e}(\mathbf{u}_\rho)\mathbf{n} = \mathbf{g} & \text{on}\;\Gamma_N \\ \mathrm{A}\,\mathbf{e}(\mathbf{u}_\rho)\mathbf{n} = 0 & \text{on}\;\partial w_\rho \cup \Gamma_0 \end{cases} \tag{13}$$

If the objective function admits the following so-called topological asymptotic expansion for small $\rho > 0$

$$L(\Omega_\rho) = L(\Omega) + \rho^2 D_T L(x_0) + o(\rho^2) \tag{14}$$

then $D_T L(x_0)$ is called the topological derivative at point x_0. The following result gives the expressions of the topological derivative for the least square error $L(\Omega)$.

$$D_T L(x) = -\frac{\pi(\lambda + 2\mu)}{2\mu(\lambda+\mu)}\left\{4\mu A\mathbf{e}(\mathbf{u})\cdot\mathbf{e}(\mathbf{p}) + (\lambda-\mu)\,\mathrm{tr}(\mathrm{A}\,\mathbf{e}(\mathbf{u}))\,\mathrm{tr}(\mathbf{e}(\mathbf{p}))\right\}(x) \tag{15}$$

where λ and μ is the Lame constant. At points x where $D_T L(x)$ is negative, we introduce holes into the current domain Ω. Since this criterion applies for infinitesimal holes, we should not remove too much material. In practice it is better to nucleate holes only at the minimum (negative) points of this topological derivative.

4. Algorithm based on Level set method

4.1 Level set method

Consider D a bounded domain in which all admissible shapes Ω are included, i.e. $\Omega \subset D$. In numerical practice, the domain D will be uniformly meshed and we parameterize the boundary of Ω using a level set function. We define this level set function ϕ in D such that

$$\begin{cases} \phi > 0 & x \in \Omega \\ \phi = 0 & x \in \partial\Omega \\ \phi < 0 & x \in D/\Omega \end{cases} \tag{16}$$

The evolution of the level set function ϕ is governed by the following Hamilton-Jacobi transport equation

$$\frac{\partial \phi}{\partial t} + V_n |\nabla \phi| = 0 \quad \text{in} \quad D, \tag{17}$$

where $V_n(t,x)$ is the normal velocity of the shape's boundary.

The choice of the normal velocity V_n is based on the shape derivative computed in section 2.2

$$\frac{\partial L}{\partial \Omega}(\mathbf{V}) = \int_{\Gamma_0} (\mathbf{A}\,\mathrm{e}(\mathbf{u}) \cdot \mathrm{e}(\mathbf{p}) - \mathbf{p} \cdot \mathbf{g}) \mathbf{V} \cdot \mathbf{n} \mathrm{ds}, \tag{18}$$

The simplest choice is to take the steepest descent $V_n = \mathbf{V} \cdot \mathbf{n} = -(\mathbf{A}\,\mathrm{e}(\mathbf{u}) \cdot \mathrm{e}(\mathbf{p}) - \mathbf{p} \cdot \mathbf{g})$ which can lead that $\frac{\partial L}{\partial \Omega}(\mathbf{V})$ is always negative.

4.2 Numerical implementation

The procedure of the identification algorithm is as follow:

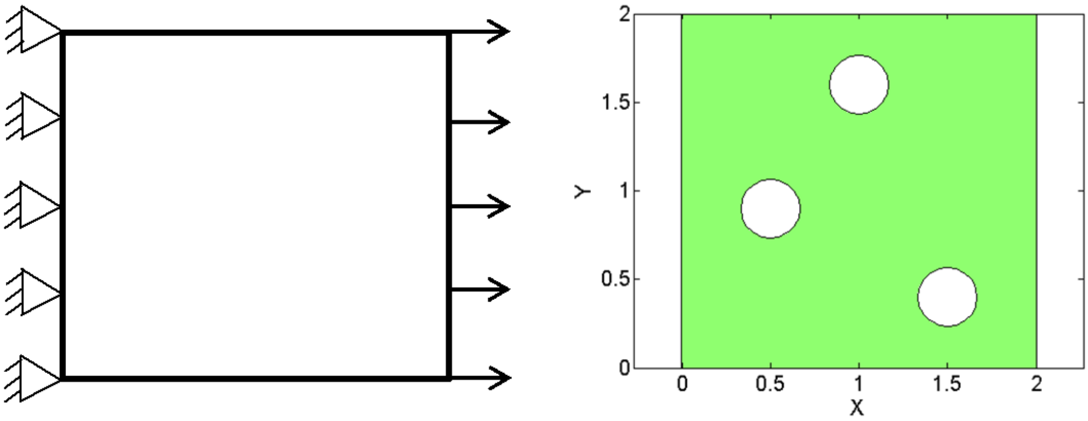
1. Initialization of the level set function corresponding to a solid design domain.

2. Iteration until convergence:

(a) Compute the topological derivative through the forward problem and the adjoint problem. Then produce voids in the problem domain corresponding to the topological derivative.

(b) Compute the shape derivative through the forward problem and the adjoint problem. Then update the level set function by taking the shape derivative as the velocity field.

5. Numerical examples



(a) (b)

Fig.1 The configuration of voids identification
(a) A plate with boundary conditions and loads (b) An objective configuration with three circular voids

In this section, examples with multiple voids are given. A plate and the boundary conditions are displayed on Fig.1 (a). The domain of plate is a square sheet of size 2×2 with the left edge fixed. A uniformly distributed unit load is applied on the right edge of plate. The plate is discretized with a rectangular 40×40 mesh. The elastic modulus of solid domain is normalized to 1, and the Poisson ratio is 0.3. The node displacements of the whole edge of the plate are measured by simulation results of FEM.

The objective configuration of plane plate with three circular voids is given in Fig.1 (b). Fig. 2 (a-e) shows the corresponding identification process. In Fig. 2 (f), the dotted line represents the objective voids that to be identified, and the solid lines represent the identified boundary which will evolve during the identification process. It can be seen that the boundary gradually converges to the objective shape with the increase of iteration number, and finally the identification voids are in good agreement with the objective voids.

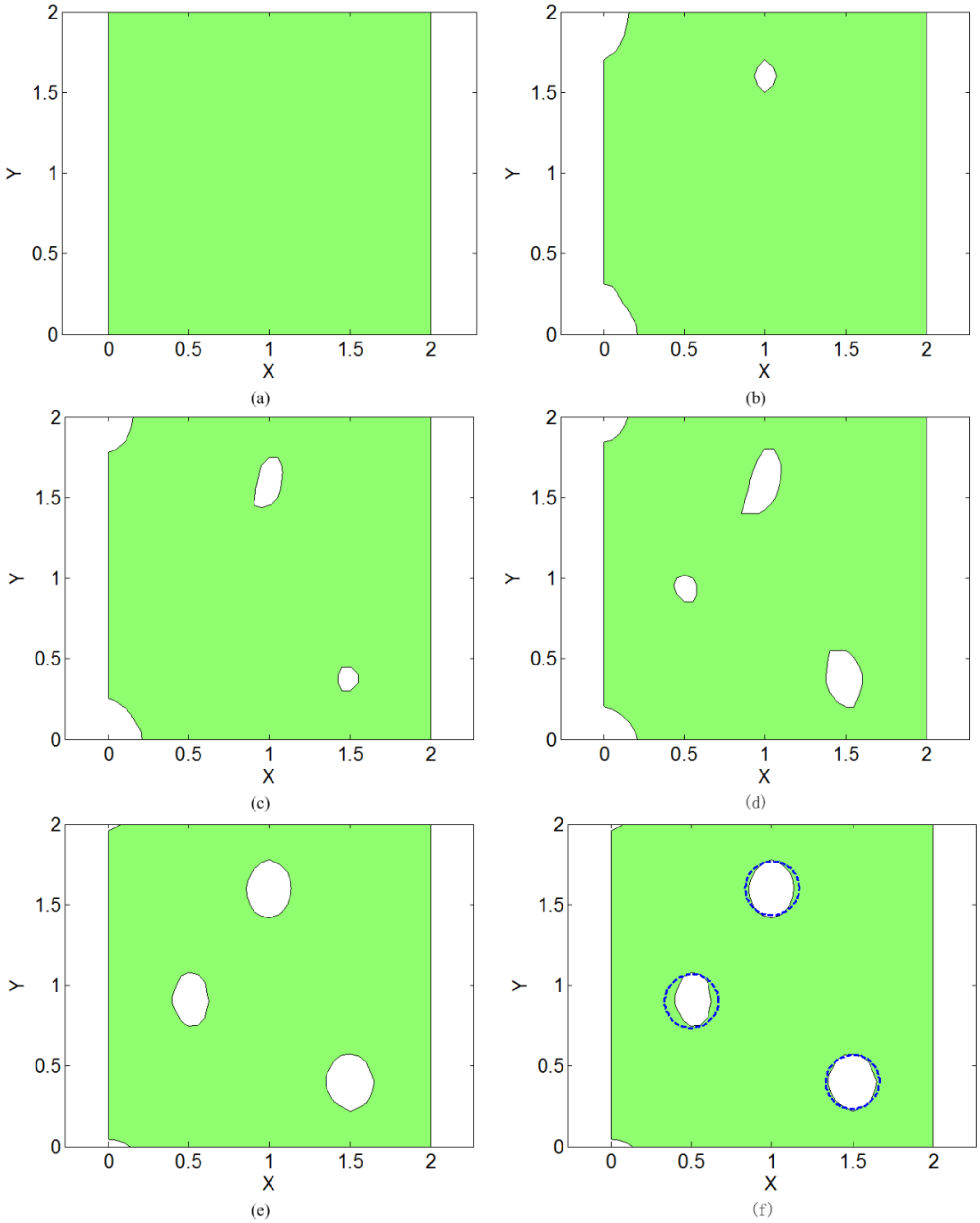


Fig.2 Iteration process of voids identification

6. Conclusions

In the present paper, an identification algorithm coupled with level set method and topological derivative is developed to identify the voids in structures. This algorithm is an optimization process. The level set method is used to represent the boundary of voids. FEM is used to solve the forward problem. The level set function is update by the velocity field which corresponds to the shape derivative of the objective function. The topological derivative of the objective function is computed to add voids to the problem domain every several iterations. From the identification results of 2D plane plate with multiple voids, it is indicated that the present algorithm can effectively identify the position of voids in structures.

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