Analytiacl Solution for Sandwiches Cantilever Beam

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Abstract:

The sandwiches cantilever beam which often appeared in composite materials science. Firstly, we give the governing equations and interface condition for solid mechanics problems from the elastic theory. Then a stress function is supposed, which can be used to describe the strain and displacement function.Finally according to the boundary conditions, we can achieve the stress and displacement solution.

Keywords: Sandwiches cantilever beam; analytical solution; strain; displacement

1. Introduction

Most problems in engineering mechanics can be described in the form of differential equation, integrals and all kinds of algebraic form. While we hope to obtain the analytical solutions for the most of the practical problems, we can only achieve the numerical solution insteads. Therefore most numerical methods have been developed, such as Finite Element Methods[1-3], Finite Difference Methods, Finite Volume Methods, and recently Meshfree Methods[4]. And the analytical solution[5-7] are always used to check the accuracy and reliability of the numerical methods. In this paper, we obatain the analytical solution of the sandwiches cantilever beam which is fixed in one end. The analytical solution can be used to as a standard problem to test the the accuracy and reliability of the numerical methods.

2. Preliminary knowledge

Based on elastic mechanics[8] the discontinuous material problem defined in domain Ω bounded by Γ_e .(Fig.1) can be described by equilibrium equation. $\sigma_{ji,j}+b_i=0$, $\Omega=\Omega^+\bigcup\Omega^-$, where σ_{ij} is the component of stress tensor and b_i is the component of body force, and Ω^+ and Ω^- are two different materials. Boundary condintions are given as follows $\sigma_{ij}n_j=\overline{t_i},(x,y)\in\Gamma_t$; $u_i=\overline{u_i},(x,y)\in\Gamma_u$ .where t_i is the vector of tractions, u_i is the vector of displacement , n_j is the unit outward normal of Ω .By the continuity of displacements and tractions on Γ_e ,we can obtain

$$[|t_i|]=t_i^+-t_i^-=0. \tag{1}$$

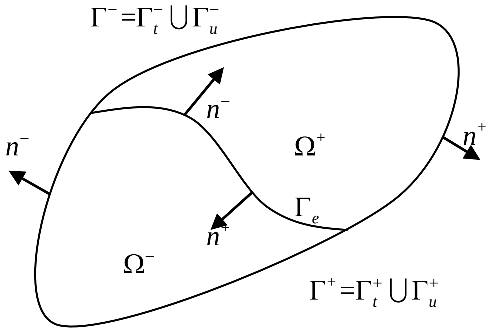
$$[|u_i|]=u_i^+-u_i^-=0. \tag{2}$$

Where $[|\cdot|]$ denotes the jump on the surface of the material,i.e. discontinuity of the Ω . By Cauchy's equation, the equation (1) can be written as

$$\sigma_{ij}{}^+n_j{}^+-\sigma_{ij}{}^-n_j{}^-=0. \tag{3}$$

Where n^+ and n^- are the unit outward normal of Ω^+ and Ω^- respectively.

Constitutive equation $\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$,where C_{ijkl} is the elasticity modulus and $\varepsilon_{kl}=\frac{1}{2}(u_{k,l}+u_{l,k})$ is the strain tensor.



3.The Expression of strain, stress and displacement functions.

Consider a two-dimensional multilayer material problem. Suppose that $\varepsilon_{ij}, \sigma_{ij}, \ \nu_i, E_i$ are the strain, stress, Poisson's ration,and Young's modulus respectively, where $i, j = 1, 2, 3$. For each layer of the multilayer material, all have the same elasticity equation form. Firstly, the static equilibrium equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{4}$$

By constitutive equations of plane problem $\sigma = D\varepsilon$,we can obtain

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \gamma_{xy} = 2\frac{1+\nu}{E}\tau_{xy}. \tag{5}$$

By $\varepsilon = Lu$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{6}$$

From the formula (6) we can get the equation of strain compatibility

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{7}$$

Let U_1, U_2, U_3 be the stresses from top to bottom in Sandwiches Cantilever Beam problem, which satisfy the following equations respctively

$$\sigma^i{}_x = \frac{\partial^2 U_i}{\partial y^2}, \sigma^i{}_y = \frac{\partial^2 U_i}{\partial x^2}, \tau^i{}_{xy} = -\frac{\partial^2 U_i}{\partial x \partial y}, i = 1, 2, 3. \tag{8}$$

By the elastic theory[9], we can obtain

$$\sigma^i{}_y = \frac{\partial^2 U_i}{\partial x^2} = f_i(y). \tag{9}$$

Integrating both sides of (9) the stress can be expressed as

$$U_i = \frac{x^2}{2} f_i(y) + x f_{i,1}(y) + f_{i,2}(y). \tag{10}$$

Substituting (10) into (8), we have

$$\sigma_x^i = \frac{\partial^2 U_i}{\partial y^2} = \frac{x^2}{2}\frac{\partial^2 f_i(y)}{\partial y^2} + \frac{\partial^2 f_{i,1}(y)}{\partial y^2} + \frac{\partial^2 f_{i,2}(y)}{\partial y^2} \tag{11}$$

$$\tau_{xy}^i = -\frac{\partial^2 U_i}{\partial x \partial y} = -x\frac{\partial f_i(y)}{\partial y} - \frac{\partial f_{i,1}(y)}{\partial y} \tag{12}$$

Substituting (9),(11),(12) into (5) we have

$$\begin{cases} \varepsilon_x^i = \frac{1}{E_i}[\frac{x^2}{2}\frac{\partial^2 f_i(y)}{\partial y^2} + x\frac{\partial^2 f_{i,1}(y)}{\partial y^2} + \frac{\partial^2 f_{i,2}(y)}{\partial y^2} - \nu_i f_i(y)], \\ \varepsilon_y^i = \frac{1}{E_i}[f_i(y) - \nu_i(\frac{x^2}{2}\frac{\partial^2 f_i(y)}{\partial y^2} + \frac{\partial^2 f_{i,1}(y)}{\partial y^2} + \frac{\partial^2 f_{i,2}(y)}{\partial y^2})], \\ \gamma_{xy}^i = 2\frac{1+\nu_i}{E_i}(-x\frac{\partial f_i(y)}{\partial y} - \frac{\partial f_{i,1}(y)}{\partial y}). \end{cases} \tag{13}$$

We need to confirm the expressions of $f_i(y)$, $f_{i,1}(y)$, $f_{i,2}(y)$ to obtain U_i, by (5) and (13) we have

$$\begin{cases} \frac{\partial^2 \varepsilon_x{}^i}{\partial y^2} = \frac{1}{E_i}[\frac{x^2}{2}\frac{\partial^4 f_i(y)}{\partial y^4} + x\frac{\partial^4 f_{i,1}(y)}{\partial y^4} + \frac{\partial^4 f_{i,2}(y)}{\partial y^4} - \nu_i\frac{\partial^2 f_i(y)}{\partial y^2}], \\ \frac{\partial^2 \varepsilon_y{}^i}{\partial x^2} = \frac{1}{E_i}(-\nu_i\frac{\partial^2 f_i(y)}{\partial y^2}), \\ \frac{\partial^2 \gamma_{xy}{}^i}{\partial x \partial y} = -2\frac{1+\nu_i}{E_i}\frac{\partial^2 f_i(y)}{\partial y^2}. \end{cases} \tag{14}$$

Substituting (14) into (7) and simplifying ,we have

$$\frac{x^2}{2}\frac{\partial^4 f_i(y)}{\partial y^4} + x\frac{\partial^4 f_{i,1}(y)}{\partial y^4} + \frac{\partial^4 f_{i,2}(y)}{\partial y^4} + 2\frac{\partial^2 f_i(y)}{\partial y^2} = 0.$$

Since $f_i(y)$, $f_{i,1}(y)$, $f_{i,2}(y)$ can be taken arbitrarily, we have

$$\frac{\partial^4 f_i(y)}{\partial y^4} = 0, \frac{\partial^4 f_{i,1}(y)}{\partial y^4} = 0, \frac{\partial^4 f_{i,2}(y)}{\partial y^4} + 2\frac{\partial^2 f_i(y)}{\partial y^2} = 0. \tag{15}$$

Integrating the first two equations of (15) we can get the expressions of $f_i(y)$, $f_{i,1}(y)$

$$f_i(y) = A_i y^3 + B_i y^2 + C_i y + D_i,\ f_{i,1}(y) = F_i y^3 + G_i y^2 + H_i y.$$

By the third equation of (15) and (16) we have

$$f_{i,2}(y) = -\frac{A_i}{10}y^5 - \frac{B_i}{6}y^4 + M_i y^3 + N_i y^2.$$

Substituting (16),(17) into (10) the stress can be expressed as

$$\begin{aligned} U_i = {} & \frac{x^2}{2}(A_i y^3 + B_i y^2 + C_i y + D_i) + x(F_i y^3 + G_i y^2 + H_i y) - \\ & \frac{A_i}{10}y^5 - \frac{B_i}{6}y^4 + M_i y^3 + N_i y^2, \end{aligned} \tag{18}$$

Where A_i, B_i, C_i, D_i, F_i, G_i, H_i, M_i, $N_i, i = 1, 2, 3$ are undetermined coefficients.

Thus $\sigma_x^i, \sigma_y^i, \tau_{xy}^i$ can be denoted as

$$\begin{cases} \sigma_x{}^i = \frac{\partial^2 U_i}{\partial y^2} = \frac{x^2}{2}(6A_i y + 2B_i) + x(6F_i + 2G_i) - 2A_i y^3 - 2B_i y^2 + 6M_i y + 2N_i, \\ \sigma_y{}^i = \frac{\partial^2 U_i}{\partial x^2} = A_i y^3 + B_i y^2 + C_i y + D_i, \\ \tau_{xy}{}^i = -\frac{\partial^2 U_i}{\partial x \partial y} = -x(3A_i y^2 + 2B_i y + C_i) - 3F_i y^2 - 2G_i y - H_i. \end{cases} \tag{19}$$

By (6) and (19), the strain can be expressed as

$$\varepsilon_x{}^i = \frac{x^2(3A_i y + B_i) + x(6F_i + 2G_i) - 2A_i y^3 - 2B_i y^2 + 6M_i y + 2N_i}{E_i} - \nu_i \frac{A_i y^3 + B_i y^2 + C_i y + D}{E_i}, \tag{20}$$

$$\varepsilon_y{}^i = \frac{A_i y^3 + B_i y^2 + C_i y + D}{E_i} - \nu_i \frac{x^2(3A_i y + B_i) + x(6F_i + 2G_i) - 2A_i y^3 - 2B_i y^2 + 6M_i y + 2N_i}{E_i}, \tag{21}$$

$$\gamma_{xy}{}^i = -\frac{2(1+\nu_i)[-x(3A_i y^2 + 2B_i y + C_i) - 3F_i y^2 - 2G_i y - H_i]}{E_i}. \tag{22}$$

Integrating both sides of the following formulae $\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}$, we have

$$u_i = \frac{-(D_i\nu_i - 2N_i - 6M_i y + 2A_i y^3 + 2B_i y^2 + C_i\nu_i y + A_i\nu_i y^3 + B_i\nu_i y^2)}{E_i} x + \frac{G_i + 3F_i y}{E_i} x^2 + \frac{B_i + 3A_i y}{3E_i} x^3 + h_i(y), \tag{23}$$

$$v_i = \left(\frac{y(D_i - 2N_i\nu_i)}{E_i} + \frac{y^4(A_i + 2A_i\nu_i)}{4E_i} + \frac{y^3(B_i + 2B_i\nu_i)}{3E_i} + \frac{y^2(C_i - 6M_i\nu_i)}{2E_i}\right)x - \frac{\nu_i y(2G_i + 3F_i y)}{E_i} x^2 - \frac{\nu_i y(2B_i + 3A_i y)}{2E_i} x^3 + g_i(x). \tag{24}$$

By the formulae $\gamma_{xy} = 2\frac{1+\nu}{E}\tau_{xy}$ and $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, we have

$$6F_i y + 4G_i y + 3\nu_i F_i y^2 + 2\nu_i G_i y + \frac{dh_i(y)}{dy} E_i + 2xG_i + x^3 A_i + 3x^2 F_i + 6M_i x + 2H_i + \nu_i C_i x + 2\nu_i H_i + \frac{dg_i(x)}{dx} E_i = 0 \tag{25}$$

where $i = 1, 2, 3$.

Since $h_i(y)$ and $g_i(x)$ can be taken arbitrarily, then suppose that

$$6F_i y + 4G_i y + 3\nu_i F_i y^2 + 2\nu_i G_i y + \frac{dh_i(y)}{dy} E_i = 0$$

$$2xG_i + x^3 A_i + 3x^2 F_i + 6M_i x + 2H_i + \nu_i C_i x + 2\nu_i H_i + \frac{dg_i(x)}{dx} E_i = 0 \tag{26}$$

Then we obtain

$$h_i(y) = -\frac{(2+\nu_i)(F_i y^3 + G_i y^2)}{E_i} + I_i y + J_i,$$

$$g_i(x) = -\frac{A_i x^4 + 4F_i x^3 + 4G_i x^2 + 12M_i x^2 + 2\nu_i C_i x^2 + 8(1+\nu_i)H_i x}{4E_i} - I_i x + L_i. \tag{27}$$

Substituting $h_i(y)$, $g_i(x)$ into (23),(24), the expression of displacement

$$u_i = \frac{-(D_i\nu_i - 2N_i - 6M_i y + 2A_i y^3 + 2B_i y^2 + C_i\nu_i y + A_i\nu_i y^3 + B_i\nu_i y^2)}{E_i}x + \frac{G_i + 3F_i y}{E_i}x^2 + \frac{B_i + 3A_i y}{3E_i}x^3 - \frac{(2+\nu_i)(F_i y^3 + G_i y^2)}{E_i} + I_i y + J, \tag{28}$$

$$v_i = [\frac{y(D_i - 2N_i\nu_i)}{E_i} + \frac{y^4(A_i + 2A_i\nu_i)}{4E_i} + \frac{y^3(B_i + 2B_i\nu_i)}{3E_i} + \frac{y^2(C_i - 6M_i\nu_i)}{2E_i} - \frac{2(1+\nu_i)H_i}{E_i} - I_i]x - [\frac{2\nu_i y(2G_i + 3F_i y) + 2G_i + 6M_i + \nu_i C_i}{2E_i}]x^2 - \frac{\nu_i y(2B_i + 3A_i y) + 2F_i}{2E_i}x^3 - \frac{A_i}{4E_i}x^4 + L_i. \tag{29}$$

4.Determine the coefficients by the boundary conditions

The following is to determine the coefficients $A_i, B_i, C_i, D_i, F_i, G_i, H_i, I_i, J_i, L_i, M_i, N_i$, where $i = 1, 2, 3$. The natural boundary conditions:

On the top layer, i.e. $y = h_1$,where $\sigma_y^1 = 0, \tau_{xy}^1 = 0.$ By the second equation of (19) we have

$$A_1 h_1^3 + B_1 h_1^2 + C_1 h_1 + D_1 = 0. \tag{30}$$

Combine the third equation of (19) we obtain

$$\tau_{xy}^1 = -x(3A_1 h_1^2 + 2B_1 h_1 + C_1) - 3F_1 h_1 - 2G_1 h_1 - H_1 = 0,$$

since the value of x is not unique,then

$$3A_1 h_1^2 + 2B_1 h_1 + C_1 = 0, \; 3F_1 h_1 + 2G_1 h_1 + H_1 = 0 \tag{31}$$

On the bottom layer, i.e. $y = -h_1$, where $\sigma_y^3 = 0, \tau_{xy}^3 = 0.$ Similarly

$$-A_3 h_1^3 + B_3 h_1^2 - C_3 h_1 + D_3 = 0, \tag{32}$$

$$3A_3(-h_1)^2 + 2B_3(-h_1) + C_3 = 0, \tag{33}$$

$$3F_3(-h_1) + 2G_3(-h_1) + H_3 = 0. \tag{34}$$

When $x = l$, the integral form of the natural boundary condition should satisfy the following

$$\int_{-h_1}^{-h_2} \sigma_x^3 dy + \int_{-h_2}^{h_2} \sigma_x^2 dy + \int_{h_2}^{h_1} \sigma_x^1 dy = 0, \tag{35}$$

$$\int_{-h_1}^{-h_2} \sigma_x^3 y dy + \int_{-h_2}^{h_2} \sigma_x^2 y dy + \int_{h_2}^{h_1} \sigma_x^1 y dy = 0, \tag{36}$$

$$\int_{-h_1}^{-h_2} \tau_{xy}^3 dy + \int_{-h_2}^{h_2} \tau_{xy}^2 dy + \int_{h_2}^{h_1} \tau_{xy}^1 dy = -q. \tag{37}$$

Substituting $\sigma_x^i, \tau_{xy}^i (i = 1, 2, 3)$ in (19) into (35)~(37), deforming obtain the equations of A_i, $B_i, F_i, G_i, M_i, N_i (i = 1, 2, 3)$.

$$\frac{l}{2}(h_2^2 - h_1^2)(3l^2 - (h_2^2 + h_1^2))A_3 + (-h_2 + h_1)(l^2 - \frac{2}{3}(h_1^2 + h_1 h_2 + h_2^2))B_3 + 3l(h_2^2 - h_1^2)F_3 + 2l(-h_2 + h_1)G_3 + 3M_3(h_2^2 - h_1^2) + 2N_3(-h_2 + h_1) + h_2(2l^2 - \frac{4}{3}h_2^2)B_2 + 4h_2 l G_2 + 4h_2 N_2 + \frac{l}{2}(h_1^2 - h_2^2)(3l^2 - (h_1^2 + h_2^2))A_1 + (h_1 - h_2)(l^2 - \frac{2}{3}(h_1^2 + h_1 h_2 + h_2^2))B_1 + 3l(h_1^2 - h_2^2)F_1 + 2l(h_1 - h_2)G_1 + 3M_1(h_1^2 - h_2^2) + 2N_1(h_1 - h_2) = 0 \tag{38}$$

$$[l^2(h_1^3-h_2^3)-\frac{2}{5}(h_1^5-h_2^5)]A_3+\frac{1}{2}(h_2^2-h_1^2)(l^2-(h_1^2+h_2^2))B_3+2l(h_1^3-h_2^3)F_3+$$
$$l(h_2^2-h_1^2)G_3+2(-h_2^3+h_1^3)M_3+(h_2^2-h_1^2)N_3+2h_2^3(l^2-\frac{2}{5}h_2^2)A_2+4lh_2^3F_2+$$
$$4h_2^3M_2+[l^2(h_1^3-h_2^3)-\frac{2}{5}(h_1^5-h_2^5)]A_1+\frac{1}{2}(h_1^2-h_2^2)(l^2-(h_1^2+h_2^2))B_1+$$
$$2l(h_1^3-h_2^3)F_1+l(h_1^2-h_2^2)G_1+2(h_1^3-h_2^3)M_1+(h_1^2-h_2^2)N_1=0. \tag{39}$$

$$l(h_1^3-h_2^3)A_3+l(h_2^2-h_1^2)B_3+l(h_1-h_2)C_3+(h_1^3-h_2^3)F_3+(h_2^2-h_1^2)G_3+$$
$$(h_1-h_2)H_3+2lh_2^3A_2+2lh_2C_2+2h_2^3F_2+2h_2H_2+l(h_1^3-h_2^3)A_1+$$
$$l(h_1^2-h_2^2)B_1+l(h_1-h_2)C_1+(h_1^3-h_2^3)F_1+(h_1^2-h_2^2)G_1+(h_1-h_2)H_1=-q. \tag{40}$$

The essential boundary conditions :

When $(x,y)=(0,0)$,we have $u_2(0,0)=0, v_2(0,0)=0, \frac{\partial v_2}{\partial x}(0,0)=0.$ By the displacement expressions (28),(29) obatin

$$J_2=0,\ L_2=0,-\frac{2(1+\nu_2)H_2}{E_2}+I_2=0. \tag{41}$$

By the continuity of displacement and stress and (2),(3) obtain

$$u_1=u_2, v_1=v_2, \sigma_y^1=\sigma_y^2, \tau_{xy}^1=\tau_{xy}^2, \text{当 } y=-h_2 \text{ 时},$$
$$u_3=u_2, v_3=v_2, \sigma_y^3=\sigma_y^2, \tau_{xy}^3=\tau_{xy}^2, \text{当 } y=h_2 \text{ 时},$$

Since $u_1=u_2$,where $y=-h_2$ then

$$\{\frac{-\left(D_1\nu_1-2N_1+6M_1h_2-2A_1h_2^3+2B_1h_2^2-C_1\nu_1h_2-A_1\nu_1h_2^3+B_1\nu_1h_2^3\right)}{E_1}+$$
$$\frac{\left(D_2\nu_2-2N_2+6M_2h_2-2A_2h_2^3+2B_2h_2^2-C_2\nu_2h_2-A_2\nu_2h_2^3+B_2\nu_2h_2^3\right)}{E_2}\}x+$$
$$(\frac{G_1-3F_1h_2}{E_1}-\frac{G_2-3F_2h_2}{E_2})x^2+(\frac{B_1-3A_1h_2}{3E_1}-\frac{B_2-3A_2h_2}{3E_2})x^3-$$
$$\frac{(2+\nu_1)(-F_1h_2^3+G_1h_2^2)}{E_1}-I_1h_2+J_1+\frac{(2+\nu_2)(-F_2h_2^3+G_2h_2^2)}{E_1}+I_2h_2-J_2=0.$$

And the value of x is not unique, we have

$$\frac{-\left(D_1\nu_1-2N_1+6M_1h_2-2A_1h_2^3+2B_1h_2^2-C_1\nu_1h_2-A_1\nu_1h_2^3+B_1\nu_1h_2^2\right)}{E_1}+$$
$$\frac{\left(D_2\nu_2-2N_2+6M_2h_2-2A_2h_2^3+2B_2h_2^2-C_2\nu_2h_2-A_2\nu_2h_2^3+B_2\nu_2h_2^2\right)}{E_2}=0, \tag{42}$$

$$\frac{G_1-3F_1h_2}{E_1}-\frac{G_2-3F_2h_2}{E_2}=0,\ \frac{B_1-3A_1h_2}{3E_1}-\frac{B_2-3A_2h_2}{3E_2}=0, \tag{43}$$

$$-\frac{(2+\nu_1)(-F_1h_2^3+G_1h_2^2)}{E_1}-I_1h_2+J_1+\frac{(2+\nu_2)(-F_2h_2^3+G_2h_2^2)}{E_1}+I_2h_2-J_2=0. \tag{44}$$

Similarly since $v_1=v_2$,when $y=-h_2$ then we have

$$-\frac{h_2(D_1-2N_1\nu_1)}{E_1}+\frac{h_2^4(A_1+A_1\nu_1)}{4E_1}-\frac{h_2^3(B_1+2B_1\nu_1)}{3E_1}+\frac{h_2^2(C_1-6M_1\nu_1)}{2E_1}-\frac{2(1+\nu_1)H_1}{E_1}-I_1+$$

$$\frac{h_2(D_2-2N_2\nu_2)}{E_2}-\frac{h_2^4(A_2+A_2\nu_2)}{4E_2}+\frac{h_2^3(B_2+2B_2\nu_2)}{3E_2}-\frac{h_2^2(C_2-6M_2\nu_2)}{2E_2}+\frac{2(1+\nu_2)H_2}{E_2}+I_2=0, \tag{45}$$

$$-\frac{-2\nu_1h_2(2G_1-3F_1h_2)+2G_1+6M_1+\nu_1C_1}{2E_1}+\frac{-2\nu_2h_2(2G_2-3F_2h_2)+2G_2+6M_2+\nu_2C_2}{2E_2}=0, \tag{46}$$

$$-\frac{-\nu_1h_2(2B_1+3A_1y)+2F_1}{2E_1}+\frac{-\nu_2h_2(2B_2+3A_2y)+2F_2}{2E_2}=0, \tag{47}$$

$$-\frac{A_1}{4E_1}+\frac{A_2}{4E_2}=0, \tag{48}$$

$$L_1-L_2=0. \tag{49}$$

From $\sigma_y^1=\sigma_y^2$ obtain

$$-(A_1-A_2)h_2^3+(B_1-B_2)h_2^2-(C_1-C_2)h_2+D_1-D_2=0. \tag{50}$$

From $\tau_{xy}^1=\tau_{xy}^2$ obtain

$$-x(3A_1h_2^2-2B_1h_2+C_1-3A_2h_2^2+2B_2h_2-C_2)-3F_1h_2^2-2G_1h_2-$$
$$H_1+3F_2h_2^2+2G_2h_2+H_2=0.$$

And the value of x is not unique, we have

$$3A_1h_2^2-2B_1h_2+C_1-3A_2h_2^2+2B_2h_2-C_2=0, \tag{51}$$

$$-3F_1h_2^2-2G_1h_2-H_1+3F_2h_2^2+2G_2h_2+H_2=0. \tag{52}$$

Similarly when $y=h_2$,since $u_3=u_2$

$$-\frac{D_3\nu_3-2N_3-6M_3h_2+2A_3h_2^3+2B_3h_2^2+C_3\nu_3h_2+A_3\nu_3h_2^3+B_3\nu_3h_2^2}{E_3}+$$
$$\frac{D_2\nu_2-2N_2+6M_2h_2+2A_2h_2^3+2B_2h_2^2+C_2\nu_2h_2+A_2\nu_2h_2^3+B_2\nu_2h_2^2}{E_2}=0, \tag{53}$$

$$\frac{G_3+3F_3h_2}{E_3}-\frac{G_2+3F_2h_2}{E_2}=0, \tag{54}$$

$$\frac{B_3+3A_3h_2}{3E_3}-\frac{B_2+3A_2h_2}{3E_2}=0. \tag{55}$$

$$-\frac{(2+\nu_2)(F_2h_2^3+G_2h_2^2)}{E_2}+I_2h_2+J_2+\frac{(2+\nu_3)(F_3h_2^3+G_3h_2^2)}{E_3}-I_3h_2-J_3=0. \tag{56}$$

since $\nu_3=\nu_2$ then

$$\frac{h_2(D_3-2N_3\nu_3)}{E_3}+\frac{h_2^4(A_3+2A_3\nu_3)}{4E_3}+\frac{h_2^3(B_3+2B_3\nu_1)}{3E_3}+\frac{h_2^2(C_3-6M_3\nu_3)}{2E_3}-\frac{2(1+\nu_3)H_3}{E_3}-I_3-$$

$$\frac{h_2(D_2-2N_2\nu_2)}{E_2}-\frac{h_2^4(A_2+A_2\nu_2)}{4E_2}-\frac{h_2^3(B_2-2B_2\nu_2)}{3E_2}-\frac{h_2^2(C_2-6M_2\nu_2)}{2E_2}+\frac{2(1+\nu_2)H_2}{E_2}+I_2=0, \tag{57}$$

$$-\frac{2\nu_3h_2(2G_3+3F_3h_2)+2G_3+6M_3+\nu_3C_3}{2E_3}+\frac{2\nu_2h_2(2G_2+3F_2h_2)+2G_2+6M_2+\nu_2C_2}{2E_2}=0, \tag{58}$$

$$-\frac{\nu_3 h_2(2B_3+3A_3 y)+2F_3}{2E_3}+\frac{\nu_2 h_2(2B_2+3A_2 h_2)+2F_2}{2E_2}=0, \quad (59)$$

$$-\frac{A_3}{4E_3}h_2^4+\frac{A_2}{4E_2}h_2^4=0, \quad (60)$$

$$L_3-L_2=0. \quad (61)$$

By $\sigma_y^1=\sigma_y^3$ and $\tau_{xy}^1=\tau_{xy}^3$ obtain

$$(A_3-A_2)h_2^3+(B_3-B_2)h_2^2+(C_3-C_2)h_2+D_3-D_2=0, \quad (62)$$

$$3A_3h_2^2+2B_3h_2+C_3-3A_2h_2^2-2B_2h_2-C_2=0, \quad (63)$$

$$3F_3h_2^2-2G_3h_2+H_3-F_2h_2^2-2G_2h_2-H_2=0. \quad (64)$$

From the above we can solve equations (30)~(34),(38)~(64) for the coefficients $A_i,B_i,C_i,D_i,F_i,G_i,H_i,I_i,J_i,L_i,M_i,N_i$, $i=1,2,3.$,then substitute them into the analytical solutions of displacement and stress as following

$$u_1(x,y)=\frac{1}{4E_2(E_2h_2^3-E_3h_2^3+h_1^3E_3)}\cdot q[6h_2E_3(h_2^2-h_1^2)(1+\nu_2)+ E_2(6(1+\nu_3)(h_2-y)h_1^2+3h_2^2y(\nu_3-\nu_2)-(4\nu_3+2\nu_2+6)h_2^3+ y^3(2+\nu_3)+3xy(2l-x))],$$

$$v_1(x,y)=\frac{-q[3(l-x)(\nu_3y+h_2^2(\nu_2-\nu_3))+x^2(3l-x)]}{4(E_2h_2^3-E_3h_2^3+h_1^3E_3)},$$

$$u_2(x,y)=\frac{yq[(1+\nu_2)(6E_3(h_2^2-h_1^2)+E_2(2y^2-6h_1^2))+3xE_2(2l-x)]}{4E_2(E_2h_2^3-E_3h_2^3+h_1^3E_3)},$$

$$v_2(x,y)=-\frac{q(3y^2\nu_2l+3x^2l-x^3-3y^2\nu_2x)}{4(E_2h_2^3-E_3h_2^3+h_1^3E_3)},$$

$$u_3(x,y)=\frac{1}{4E_2(E_2h_2^3-E_3h_2^3+h_1^3E_3)}\cdot q[E_2(h_2+y)(\nu_3(y(y-h_2)+4h_2^2-6h_1^2)-6h_1^2)+ E_2(6xyl-3x^2y+2y^3)-6E_3h_2(1+\nu_2)(h_2^2-h_1^2)-E_2h_2^2(\nu_2(3y-2h_2)-6h_2)],$$

$$v_3(x,y)=-\frac{q[3(l-x)(\nu_3y+h_2^2(\nu_2-\nu_3))+x^2(3l-x)]}{4(E_2h_2^3-E_3h_2^3+h_1^3E_3)},$$

In the top layer $[h_1,h_2]\times[0,L]$,the analytical solution of stress

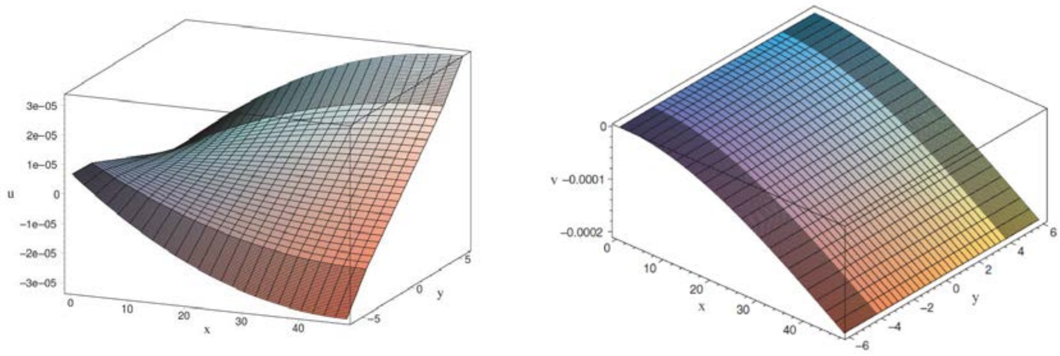
$$\sigma_x^1=\frac{3}{2}\frac{(l-x)yE_3q}{E_2h_2^3-E_3h_2^3+h_1^3E_3},\ \sigma_y^1=0,\ \tau_{xy}^1=\frac{3}{4}\frac{(y^2-h_1^2)E_3q}{E_2h_2^3-E_3h_2^3+h_1^3E_3}.$$

In the top layer $[h_2,-h_2]\times[0,L]$,the analytical solution of stress

$$\sigma_x^2=\frac{3}{2}\frac{(l-x)yE_2q}{E_2h_2^3-E_3h_2^3+h_1^3E_3},\ \sigma_y^2=0,\ \tau_{xy}^2=\frac{3}{4}\frac{(-h_2^2E_2+y^2E_2+h_2^2E_3-E_3h_1^2)q}{E_2h_2^3-E_3h_2^3+h_1^3E_3}.$$

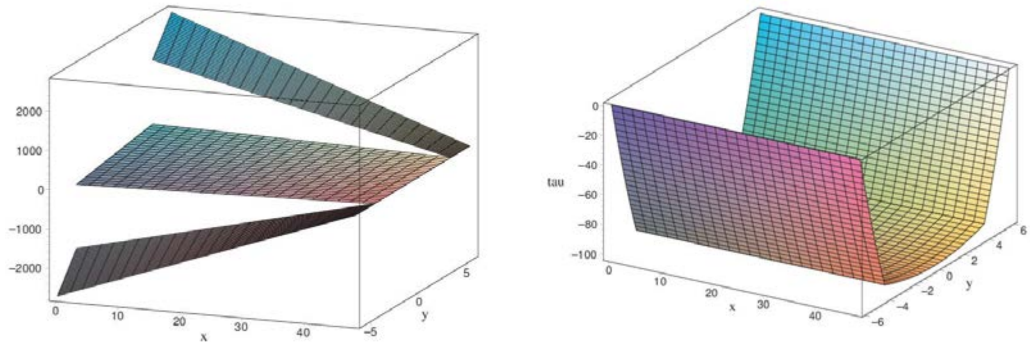
In the bottom layer $[-h_2,-h_1]\times[0,L]$,the analytical solution of stress

$$\sigma_x^3=\frac{3}{2}\frac{(l-x)yE_3q}{E_2h_2^3-E_3h_2^3+h_1^3E_3},\ \sigma_y^3=0,\ \tau_{xy}^3=\frac{3}{4}\frac{(y^2-h_1^2)E_3q}{E_2h_2^3-E_3h_2^3+h_1^3E_3}.$$



(a) (b)

Fig.2 Displacement distribution for sandwich beam (a) in x-direction (b) y-irection.



(a) (b)

Fig.3 Distribution of stress for sandwich beam (a) normal stress σ_x ans (b) shear stress τ_{xy} .

5.Conclusions

This paper focus on the analytical solution of the sandwiches cantilever beam. Firstly, we obtain the governing equations and the parametric expressions of stress and displacement by the elastic theory. Then by the interface condition and the boundary conditions we can achieve the analytical expression of stress and displacement.

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