An efficient class of fourth-order Jarratt-type methods for nonlinear equations

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Abstract

In this paper, we present a new two-point fourth-order Jarratt-type scheme based on Hansen-Patrick's family for solving nonlinear equations numerically. In terms of computational cost, each method of the proposed class requires only three functional evaluations (viz. one evaluation of function and two first-order derivatives) per full step to achieve optimal fourth-order convergence. Moreover, the local convergence analysis of the proposed methods is also given using hypotheses only on the first derivative and Lipschitz constants. Furthermore, the proposed scheme can also determine the complex zeros without having to start from a complex initial guess as would be necessary with other methods. Numerical examples and comparisons with some existing methods are included to confirm the theoretical results and high computational efficiency.

Keywords: Nonlinear equations, Jarratt-type methods, Kung-Traub conjecture, Local Convergence

Introduction

The construction of fixed point iterative methods for approximating simple zeros of a real valued function is an important task in theory and practice, not only in applied mathematics, but also for many applied scientific branches. In this paper, we consider iterative methods for solving a nonlinear equation of the form

$$f(x) = 0, \tag{1}$$

where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function defined on an open interval D. Analytical methods for solving such equations are almost non-existent and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iterative procedure. One of the most famous and basic tool for solving such equations is the Newton's method [18] given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n \geq 0$. It converges quadratically for simple roots and linearly for multiple roots. In order to improve its local order of convergence, many higher-order methods have been proposed and analyzed in [1, 4, 19]. One such well-known scheme is the classical cubically convergent Hansen-Patrick's family [6] defined by

$$x_{n+1} = x_n - \left[\frac{\beta + 1}{\beta \pm \{1 - (\beta + 1)L_f(x_n)\}^{1/2}}\right] \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

where $L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'^2(x_n)}$ and $\beta \in \mathbb{R}\backslash\{-1\}$. This family includes Ostrowski's square-root method for ($\beta = 0$), Euler's method for ($\beta = 1$), Laguerre's method for $\left(\beta = \frac{1}{\nu-1}, \nu \neq 1\right)$ and as a limiting case, Newton's method. Despite the cubic convergence, this scheme is considered less practical from a computational point of view because of the expensive second-order derivative evaluation. This fact motivated many researchers to investigate the idea of developing multipoint iterative methods for solving nonlinear equations.

Multipoint iterative methods [13, 14] for solving nonlinear equations are of great practical importance since they circumvent the limitations of one-point methods regarding the convergence order and computational efficiency. The main objective in the construction of the new iterative methods is to obtain the maximal computational efficiency. In other words, the aim is to attain convergence order as high as possible with fixed number of functional evaluation per iteration. According to the Kung-Traub conjecture [18], the order of convergence of any multipoint method without memory requiring d functional evaluations per iteration, cannot exceed the bound 2^{d-1}, called the optimal order. Consequently, convergence order of an optimal iterative method without memory consuming three functional evaluations cannot exceed four. Also, efficiency of an iterative method is measured by the efficiency index [18] defined as $E = p^{\frac{1}{d}}$, where p is the order of convergence. King's family [10], Ostrowski's method [18] and Jarratt's method [7, 12] are the well-known fourth-order multipoint methods without memory. The Jarratt method is widely considered and applied for its computational efficiency.

The fourth-order Jarratt's method which uses one evaluation of the function and two evaluations of the first derivatives is defined by

$$x_{n+1} = x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \tag{3}$$

where $J_f(x_n) = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}$ and $y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$. It satisfies the following error equation

$$e_{n+1} = \left(c_2^3 - c_2c_3 + \frac{c_4}{9}\right)e_n^4 + O(e_n^5).$$

Recently, Soleymani et al. [16] proposed two-point fourth-order Jarratt-type methods for obtaining simple roots of nonlinear equations, which are defined as follow:

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{2f(x_n)}{f'(x_n)+f'(y_n)}\left[\left(1 + \left(\dfrac{f(x_n)}{f'(x_n)}\right)^3\right)\left(2 - \dfrac{7}{4}\dfrac{f'(y_n)}{f'(x_n)} + \dfrac{3}{4}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^2\right)\right], \end{cases} \tag{4}$$

and its error equation is given by

$$e_{n+1} = \frac{1}{9}\left(-9 + 33c_2^3 - 9c_2c_3 + c_4\right)e_n^4 + O(e_n^5),$$

and

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{2}\left[\dfrac{1}{f'(x_n)} + \dfrac{1}{f'(y_n)}\right]\left[\left(1 + \left(\dfrac{f(x_n)}{f'(x_n)}\right)^4\right)\left(1 - \dfrac{1}{4}\left(\dfrac{f'(y_n)}{f'(x_n)} - 1\right)\right.\right. \\ \qquad \left.\left. + \dfrac{1}{2}\left(\dfrac{f'(y_n)}{f'(x_n)} - 1\right)^2\right)\right], \end{cases} \tag{5}$$

where it satifies the following error equation

$$e_{n+1} = \left(\frac{79}{27}c_2^3 - c_2c_3 + \frac{c_4}{9}\right) e_n^4 + O(e_n^5).$$

In [9], Khattri and Abbasbandy proposed an optimal fourth-order variant of Jarratt's method using one function evaluation and two first-order derivatives, which is defined as follows:

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \left[1 + \dfrac{21}{8}\dfrac{f'(y_n)}{f'(x_n)} - \dfrac{9}{2}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^2 + \dfrac{15}{8}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^3\right]\dfrac{f(x_n)}{f'(x_n)}. \end{cases} \tag{6}$$

It satisfies the following error equation

$$e_{n+1} = \frac{1}{9}\left(85c_2^3 - 9c_2c_3 + c_4\right) e_n^4 + O(e_n^5).$$

In this work, we are interested in designing a new two-point fourth-order class of iterative methods from a view point of Hansen-Patrick type methods, which does not require any second-order derivative evaluation for obtaining simple roots of nonlinear equations. Each method requires only one evaluation of the given function and two evaluations of the first-order derivative per iteration. It is also observed that the body structures of our proposed methods are simpler than the existing two-point fourth-order methods mentioned above. We also present the local convergence analysis of the proposed methods using hypotheses only on the first-order derivative and Lipschitz constants. Moreover, it is shown by way of illustration that the proposed schemes can determine the complex zeros without having to start from a complex number as would be necessary with other methods. It can be easily seen that the proposed schemes are highly efficient in multi-precision computing environment.

Uni-parametric family of Jarratt-type methods

In this section, we intend to develop a new optimal class of fourth-order Hansen-Patrick type methods, not requiring the computation of second-order derivative. For this purpose, let

$$y_n = x_n - \alpha u(x_n), \tag{7}$$

where $u(x_n) = \frac{f(x_n)}{f'(x_n)}$ and α is non-zero real parameter. Now, expanding $f'(y_n) = f'(x_n - \alpha u(x_n))$ about a point $x = x_n$ by Taylor series expansion, we have $f'(y_n) \approx f'(x_n) - \alpha u(x_n)f''(x_n)$, which further yields

$$f''(x_n) \approx \frac{f'(x_n) - f'(y_n)}{\alpha u(x_n)}. \tag{8}$$

Using this approximate value of $f''(x_n)$ in formula (2), and using weight function technique in the second step, we obtain a modified family of methods free from second-order derivative as follows:

$$\begin{cases} y_n = x_n - \alpha \dfrac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R}\backslash\{0\}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \left[\dfrac{\beta + 1}{\beta + \left\{ 1 - (\beta+1) L_f^*(x_n) \right\}^{\frac{1}{2}}} \right] H(\tau), \end{cases} \tag{9}$$

where $\beta \in \mathbb{R}$, $L_f^*(x_n) = \frac{f'(x_n) - f'(y_n)}{\alpha f'(x_n)}$ and $H : \mathbb{R} \to \mathbb{R}$ is a real variable weight function with $\tau = \frac{f'(y_n)}{f'(x_n)} = 1 + O(e_n)$. Theorem 1 illustrates that under what conditions on weight function, convergence order of the family (9) will arrive at the optimal level four.

Convergence analysis

Theorem 1 *Assume that function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is sufficiently differentiable and has a simple zero $x^* \in D$. If an initial guess x_0 is sufficiently close to $x^* \in D$, then the iterative scheme defined by* (9) *has optimal fourth-order convergence when*

$$\alpha = \frac{2}{3}, \ H(1) = 1, \ H'(1) = 0, \ H''(1) = -\frac{9}{16}(\beta - 1), \ |H'''(1)| < \infty, \tag{10}$$

where $\beta \in \mathbb{R}$. It satisfies the following error equation

$$e_{n+1} = \left[\Big(2 + \frac{32}{81} H'''(1) - \frac{3\beta}{2} - \frac{\beta^2}{2} \Big) c_2^3 - c_2 c_3 + \frac{c_4}{9} \right] e_n^4 + O(e_n^5). \tag{11}$$

Proof Let $e_n = x_n - x^*$ be the error at n^{th} iteration and $c_k = \dfrac{1}{k!} \dfrac{f^{(k)}(x^*)}{f'(x^*)}, \ k = 2, 3, \ldots$. Taking into account that $f(x^*) = 0$, we can expand $f(x_n)$ and $f'(x_n)$ about $x_n = x^*$ with the help of Taylor's series expansion. Therefore, we get

$$f(x_n) = f'(x^*) \left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5) \right), \tag{12}$$

and

$$f'(x_n) = f'(x^*) \left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5) \right). \tag{13}$$

From (12) and (13), we obtain

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + \left(2c_2^2 - 2c_3 \right) e_n^3 + \left(-4c_2^3 + 7c_2 c_3 - 3c_4 \right) e_n^4 + O(e_n^5). \tag{14}$$

Using (14) in the first step of (9), we get

$$\begin{aligned} y_n = x_n - \alpha \frac{f(x_n)}{f'(x_n)} &= (1-\alpha) e_n + \alpha c_2 e_n^2 - 2 \left(\alpha \left(c_2^2 - c_3 \right) \right) e_n^3 + \alpha \left(4c_2^3 - 7c_2 c_3 + 3c_4 \right) e_n^4 \\ &\quad + O(e_n^5). \end{aligned} \tag{15}$$

Again, by using the Taylor series, we can easily get the following expansion of $f'(y_n)$ around simple zero x^*:

$$\begin{aligned} f'(y_n) = f'(x^*) \Big(& 1 + 2(1-\alpha) c_2 e_n + \left(2\alpha c_2^2 + 3(1-\alpha)^2 c_3 \right) e_n^2 + \Big(-4\alpha c_2 \left(c_2^2 - c_3 \right) \\ & + 6(1-\alpha)\alpha c_2 c_3 + 4(1-\alpha)^3 c_4 \Big) e_n^3 + \Big(3 \left(\alpha^2 c_2^2 - 4(1-\alpha)\alpha \left(c_2^2 - c_3 \right) \right) c_3 \\ & + 12(1-\alpha)^2 \alpha c_2 c_4 + 2\alpha c_2 \left(4c_2^3 - 7c_2 c_3 + 3c_4 \right) + 5(1-\alpha)^4 c_5 \Big) e_n^4 + O(e_n^5) \Big). \end{aligned} \tag{16}$$

Using equations (13) and (16), we obtain

$$\left[\frac{\beta+1}{\beta+\left(1-\frac{(\beta+1)(f'(x_n)-f'(y_n))}{\alpha f'(x_n)}\right)^{1/2}}\right]\frac{f(x_n)}{f'(x_n)} = e_n + \frac{1}{2}\left(-c_2^2+\beta c_2^2+2c_3-3\alpha c_3\right)e_n^3 + \frac{1}{2}\Big(2c_2^3-3\beta c_2^3+\beta^2c_2^3-6c_2c_3 +6\alpha c_2c_3+6\beta c_2c_3-3\alpha\beta c_2c_3 +6c_4-12\alpha c_4+4\alpha^2c_4\Big)e_n^4+O(e_n^5), \tag{17}$$

and

$$\tau=\frac{f'(y_n)}{f'(x_n)}=1-2\left(\alpha c_2\right)e_n+3\left(2\alpha c_2^2-2\alpha c_3+\alpha^2c_3\right)e_n^2-4\Big(4\alpha c_2^3-7\alpha c_2c_3 +3\alpha^2c_2c_3+3\alpha c_4-3\alpha^2c_4+\alpha^3c_4\Big)e_n^3+O(e_n^4). \tag{18}$$

Since, it is clear from (18) that $\tau-1$ is of order $O(e_n)$. Therefore, we can expand the weight function $H(\tau)$ in the neighborhood of one using Taylor series expansion up to third-order terms as follows:

$$H(\tau)=H(1)+H'(1)\tau+\frac{1}{2!}H''(1)\tau^2+\frac{1}{3!}H'''(1)\tau^3+O(\tau^4). \tag{19}$$

Using equations (12)-(19) in scheme (9), we obtain the following error equation

$$e_{n+1}=(1-H(1))e_n+2\alpha H'(1)c_2e_n^2+\sum_{s=3}^{4}R_se_n^s, \tag{20}$$

where $R_s=R_s\left(c_2,c_3,c_4,\alpha,\beta,H(1),H^i(1)\right)$ for $i=1,2,3$.
From (20), it is clear that by inserting the following values:

$$H(1)=1,\ H'(1)=0, \tag{21}$$

we obtain atleast third-order convergence. Further, using (21) into $R_3=0$, we obtain two independent relations as follows:

$$4\alpha^2H''(1)+\beta-1=0,\quad 3\alpha-2=0, \tag{22}$$

which implies

$$\alpha=\frac{2}{3},\ H''(1)=-\frac{9}{16}\left(\beta-1\right). \tag{23}$$

Finally, using the above equations (21), (23) in (20), we obtain the following error equation

$$e_{n+1}=\left(\Big(2+\frac{32}{81}H'''(1)-\frac{3\beta}{2}-\frac{\beta^2}{2}\Big)c_2^3-c_2c_3+\frac{c_4}{9}\right)e_n^4+O(e_n^5).$$

This reveals that the modified family of Hansen-Patrick type methods (9) attains fourth-order convergence requiring only three functional evaluations, viz., $f(x_n),\ f'(x_n)$ and $f'(y_n)$, per step.

Finally, by using (10) in (19), we get

$$H(\tau) = 1 - \frac{1}{2!}\left\{\frac{9}{16}(\beta - 1)\right\}\tau^2 + \frac{1}{3!}H'''(1)\tau^3, \quad \tau = \frac{f'(y_n)}{f'(x_n)}. \tag{24}$$

For the sake of simplicity, we take $H'''(1) = 0$ and get a wide general class of Hansen-Patrick type methods defined by

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}\left[\dfrac{\beta + 1}{\beta + \left\{1 - \frac{(\beta+1)(f'(x_n) - f'(y_n))}{\alpha f'(x_n)}\right\}^{\frac{1}{2}}}\right]\left[1 - \dfrac{9(\beta - 1)}{32}\left(\dfrac{f'(y_n)}{f'(x_n)}\right)^2\right]. \end{cases} \tag{25}$$

It satisfies the following error equation

$$e_{n+1} = \left(\left(2 - \frac{3\beta}{2} - \frac{\beta^2}{2}\right)c_2^3 - c_2c_3 + \frac{c_4}{9}\right)e_n^4 + O(e_n^5). \tag{26}$$

It is interesting to note that for $\beta = 1$ in (25), we get optimal fourth-order method proposed by Kou [11].

Special cases

In this section, we discuss some interesting special cases of our proposed scheme (9) based on different forms of weight function $H(\tau)$. In the forementioned cases, it can be easily checked that weight function $H(\tau)$ satisfies all the conditions of Theorem 1.

Case 1. Let us consider the following weight function

$$H(\tau) = \frac{1}{1 + \delta_1(\tau - 1)^2}, \tag{27}$$

where $\delta_1 = \frac{9}{32}(\beta - 1)$.

It is straight forward to see from above that the weight function has one free disposable parameter, namely β. Therefore, for different particular values of β, we get various optimal fourth-order Hansen-Patrick type methods but some of the important cases are described in Table 1.

Case 2. Now, we consider the following weight function

$$H(\tau) = \frac{1 + \delta_3(\tau - 1) + \delta_1(\tau - 1)^2}{1 + \delta_4(\tau - 1) + 2\delta_1(\tau - 1)^2}, \tag{28}$$

where δ_1 is defined by (27) and δ_3, δ_4 are free disposable parameters.

Particular sub-case of (28)*:*

For $\delta_3 = \frac{3}{2}$ and $\delta_4 = \frac{1}{2}$, weight function reads as:

$$H(\tau) = \frac{32 + 48(\tau - 1) + (25 - 9\beta)(\tau - 1)^2}{16(2 + 3(\tau - 1) + (\tau - 1)^2)}. \tag{29}$$

Similarly, by varying free parameter β, we obtain various cases but some of the important cases are displayed in Table 2.

Case 3. Now, we consider the following weight function

$$H(\tau) = \frac{1+\delta_2(\tau-1)}{1+\delta_2(\tau-1)+\delta_1(\tau-1)^2}, \tag{30}$$

where δ_1 is defined by (27) and δ_2 is any free disposable parameter.
Particular sub-case of (30)*:*
For $\delta_2 = 1$, weight function reads as:

$$H(\tau) = \frac{\tau}{\tau+\delta_1(\tau-1)^2}. \tag{31}$$

Hence, by varying free parameter δ_1, one can get several different cases but some of the important cases are displayed in Table 3.

Table 1: Sub-cases of weight function (27) **and their error equations**

S.No	Particular values of β	Sub-cases and their error equations
1.	$\beta = 0$	$H(\tau) = \frac{1}{1-\frac{9}{32}(\tau-1)^2}$,
	(Ostrowski's square-root type)	$e_{n+1} = (2c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.
2.	$\beta = \frac{1}{2}$	$H(\tau) = \frac{1}{1-\frac{9}{64}(\tau-1)^2}$,
	(Laguerre's type)	$e_{n+1} = (\frac{9}{8}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.
3.	$\beta = \frac{3}{4}$	$H(\tau) = \frac{1}{1-\frac{9}{128}(\tau-1)^2}$,
	(Laguerre's type)	$e_{n+1} = (\frac{19}{32}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.

Table 2: Sub-cases of weight function (29) **and their error equations**

S.No	Particular values of β	Sub-cases and their error equations
1.	$\beta = 0$	$H(\tau) = H(\tau) = \frac{1+\frac{3(\tau-1)}{2}+\frac{25(\tau-1)^2}{32}}{1+(\frac{3}{2}+\frac{(\tau-1)}{2})(\tau-1)}$,
	(Ostrowski's square-root type)	$e_{n+1} = (c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.
2.	$\beta = \frac{1}{2}$	$H(\tau) = H(\tau) = \frac{1+\frac{3(\tau-1)}{2}+\frac{41(\tau-1)^2}{64}}{1+(\frac{3}{2}+\frac{(\tau-1)}{2})(\tau-1)}$,
	(Laguerre's type)	$e_{n+1} = (\frac{5}{8}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.
3.	$\beta = \frac{3}{4}$	$H(\tau) = H(\tau) = \frac{1+\frac{3(\tau-1)}{2}+\frac{73(\tau-1)^2}{128}}{1+(\frac{3}{2}+\frac{(\tau-1)}{2})(\tau-1)}$,
	(Laguerre's type)	$e_{n+1} = (\frac{11}{32}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.

Table 3: Sub-cases of weight function (31) **and their error equations**

S.no	Particular values of β	Sub-cases and their error equations
1.	$\beta = 0$	$H(\tau) = \frac{\tau}{\tau - \frac{9}{32}(\tau-1)^2}$,
	(Ostrowski's square-root type)	$e_{n+1} = (\frac{4}{3}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.
2.	$\beta = \frac{1}{2}$	$H(\tau) = \frac{\tau}{\tau - \frac{9}{64}(\tau-1)^2}$,
	(Laguerre's type)	$e_{n+1} = (\frac{19}{24}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.
3.	$\beta = \frac{3}{4}$	$H(\tau) = \frac{\tau}{\tau - \frac{9}{128}(\tau-1)^2}$,
	(Laguerre's type)	$e_{n+1} = (\frac{41}{96}c_2^3 - c_2c_3 + \frac{c_4}{9})e_n^4 + O(e_n^5)$.

Local convergence

The local convergence analysis of method (9) was based in the previous sections on Taylor expansions and hypotheses reaching atleast the fifth derivative of function f. These hypotheses restrict the applicability of method (9).

As a motivational example, define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \ f'(1) = 3, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously, function f''' is unbounded on D. Notice that, in particular there is a plethora of iterative methods for approximating solutions of nonlinear equations. These results show that if initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^*, the initial guess x_0 should be? These local results give no information on the radius of convergence ball for the corresponding method. We address this question for method (9). The same technique can be used to other methods.

In particular, we use only hypotheses on the first derivative to show the local convergence of method (9) and Lipschitz constants.

Let $L_0 > 0$, $L > 0$, $M \geq 1$, $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta > 0$ be given parameters. Define function g_1 on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt + 2|1-\alpha|M}{2(1 - L_0 t)},$$

and parameters r_1 and r_A by

$$r_1 = \frac{2(1 - |1-\alpha|M)}{2L_0 + L}$$

and

$$r_A = \frac{2}{2L_0 + L}.$$

Suppose that

$$M|1 - \alpha| < 1. \tag{32}$$

By (32) and the preceding definitions

$$0 < r_1 < r_A < \frac{1}{L_0},$$

$g_1(r_1) = 1$ and $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$.

Moreover, define functions p and h_p on the interval $[0, \frac{1}{L_0})$ by

$$p(t) = |\frac{\beta + 1}{\alpha}| \frac{L_0(1 + g_1(t))t}{1 - L_0 t}$$

and

$$h_p(t) = p(t) - 1.$$

We have that $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. It follows from intermediate value theorem that function h_p has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_p the smallest such zero.

Let $\varphi : [0, \frac{1}{L_0}) \to [0, +\infty)$ be a continuous function. Furthermore, define functions g_2 and h_2 on the interval $[0, \frac{1}{L_0})$ by

$$g_2(t) = \frac{1}{2(1 - L_0 t)} \Big(Lt + \frac{2L_0 M(1 + \beta)(1 + g_1(t))t}{\beta|\alpha|(1 - L_0 t)} + \frac{2M(1 + \beta)}{\beta} \varphi(t) \Big) t,$$

and $h_2(t) = g_2(t) - 1$.

Then, again we have that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. Denote by r_2 the smallest zero of function h_2 in the interval $(0, \frac{1}{L_0})$. Set

$$r = min\{r_1, r_p, r_2\}. \tag{33}$$

Then, we have that

$$0 < r < r_A, \tag{34}$$

$$0 \leq g_1(t) < 1, \tag{35}$$

$$0 \leq p(t) < 1, \tag{36}$$

and

$$0 \leq g_2(t) < 1 \tag{37}$$

for each $t \in [0, r)$.

Let $U(w, \rho)$ and $\bar{U}(w, \rho)$ denote, respectively the open and closed balls in $\mathbb{R}$ with center $w \in \mathbb{R}$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (9) using the preceding notation.

Theorem 2 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that there exist $x^* \in D$ and $L_0 > 0$ such that for each $x \in D$*

$$f(x^*) = 0, \;\; f'(x^*) \neq 0, \tag{38}$$

and

$$|f'(x^*)^{-1}(f'(x)-f'(x^*))| \leq L_0|x-x^*|. \tag{39}$$

Moreover, suppose that for each $x, y \in D_1 := D \cap U\left(x^*, \frac{1}{L_0}\right)$ *there exist* $\alpha \in \mathbb{R}\backslash\{0\}, L > 0, \beta > 0, M \geq 1$ *and a continuous nondecreasing function* $\varphi : [0, \frac{1}{L_0}) \to [0, +\infty)$ *such that*

$$M|1-\alpha| < 1,$$

$$|f'(x^*)^{-1}(f'(x)-f'(y))| \leq L|x-y|, \tag{40}$$

$$|f'(x^*)^{-1}f'(x)| \leq M, \tag{41}$$

$$|H(\tau)-1| \leq \varphi(|x-x^*|), \tag{42}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{43}$$

where the radius of convergence r *is defined by* (33) *and* $\tau = f'(x)^{-1}f'(x-\alpha f'(x)^{-1}f(x))$. *Then, the sequence* $\{x_n\}$ *generated for* $x_0 \in U(x^*, r)\backslash\{x^*\}$ *by method* (9) *is well defined, remains in* $U(x^*, r)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to the solution* x^*. *Moreover, the following estimates hold*

$$|y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| \leq |x_n - x^*| < r \tag{44}$$

and

$$|x_{n+1} - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| \leq |x_n - x^*|, \tag{45}$$

where the "g" functions are defined previously. Furthermore, for $q \in [r, \frac{2}{L_0})$, *the limit point* x^* *is the only solution of equation* $f(x) = 0$ *in* $D_2 := D \cap U(x^*, q)$.

Proof The estimates (44) and (45) shall be shown using mathematical induction. By hypothesis $x_0 \in U(x^*, r)$-$\{x^*\}$ and (39), we have that

$$|f'(x^*)^{-1}(f'(x_0)-f'(x^*))| \leq L_0|x_0-x^*| < L_0 r < 1. \tag{46}$$

Using (46) and the Banach lemma on invertible functions [2, 3, 15, 17, 18], we get that $f'(x_0) \neq 0$,

$$|f'(x_0)^{-1}f'(x^*)| \leq \frac{1}{1-L_0|x_0-x^*|} \tag{47}$$

and y_0 is well defined by the first substep of method (9) for $n = 0$. We can write by (38) that

$$f(x_0) = f(x_0) - f(x^*) = \int_0^1 f'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{48}$$

Notice that $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Then, by (41) and (48), we get that

$$|f'(x^*)^{-1}f(x_0)| \leq M|x_0 - x^*|. \tag{49}$$

Then, using (33), (35), (38), (40), (47) and (49), we obtain in turn that

$$\begin{aligned}
|y_0 - x^*| &= |x_0 - x^* - f'(x_0)^{-1}f(x_0) + (1-\alpha)f'(x_0)^{-1}f(x_0)| \\
&\leq |f'(x_0)^{-1}f'(x^*)|\left|\int_0^1 f'(x^*)^{-1}\Big(f'(x^*+\theta(x_0-x^*)) - f'(x_0)\Big)(x_0-x^*)d\theta\right| \\
&+ |1-\alpha||f'(x_0)^{-1}f'(x^*)||f'(x^*)^{-1}f(x_0)| \\
&\leq \frac{L|x_0-x^*|^2}{2(1-L_0\|x_0-x^*\|)} + \frac{|1-\alpha|M|x_0-x^*|}{1-L_0|x_0-x^*|} \\
&= g_1(|x_0-x^*|)|x_0-x^*| \leq |x_0-x^*| < r,
\end{aligned} \tag{50}$$

which shows (44) for $n = 0$ and $y_0 \in U(x^*, r)$. By (33), (36), (47) and (50), we have that

$$
\begin{aligned}
|(\beta+1)L_f^*(x_0)| &\le |\frac{\beta+1}{\alpha}|\frac{1}{1-L_0|x_0-x^*|} \\
&\Big[|f'(x^*)^{-1}(f'(x_0)-f'(x^*))| + |f'(x^*)^{-1}(f'(y_0)-f'(x^*))|\Big] \\
&\le |\frac{\beta+1}{\alpha}|\frac{L_0\Big(|x_0-x^*|+|y_0-x^*|\Big)}{1-L_0|x_0-x^*|} \\
&\le |\frac{\beta+1}{\alpha}|\frac{L_0(1+g_1(|x_0-x^*|))|x_0-x^*|}{1-L_0|x_0-x^*|} \\
&= p(|x_0-x^*|) < p(r) < 1.
\end{aligned} \tag{51}
$$

In view of (51), $1-(\beta+1)L_f^*(x_0) \ge 0$. Hence, x_1 is well defined by the second substep of method (9) for $n = 0$. Then, we can write

$$
\begin{aligned}
x_1 - x^* = x_0 - x^* - f'(x_0)^{-1}f(x_0) + f'(x_0)^{-1}f(x_0)\Big[1 - \frac{\beta+1}{\beta+\sqrt{1-(\beta+1)L_f^*(x_0)}}\Big] \\
+ f'(x_0)^{-1}f(x_0)\Big[H(\tau)-1\Big]\frac{\beta+1}{\beta+\sqrt{1-(\beta+1)L_f^*(x_0)}}.
\end{aligned} \tag{52}
$$

Then, using (33), (37), (42), (47), (49), (50), (51), (52) and the triangle inequality, we get in turn that

$$
\begin{aligned}
|x_1 - x^*| &\le |x_0 - x^* - f'(x_0)^{-1}f(x_0)| + |f'(x_0)^{-1}f'(x^*)||f'(x^*)^{-1}f(x_0)| \\
&\quad |\frac{(\beta+1)L_f^*(x_0)}{(\beta+\sqrt{1-(\beta+1)L_f^*(x_0)})(1+\sqrt{1-(\beta+1)L_f^*(x_0)})}| \\
&\quad + |f'(x_0)^{-1}f'(x^*)||f'(x^*)^{-1}f(x_0)||\frac{(\beta+1)}{\beta+\sqrt{1-(\beta+1)L_f^*(x_0)}}|\varphi(|x_0-x^*|) \\
&\le \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)} + \frac{ML_0(1+\beta)(1+g_1(|x_0-x^*|))|x_0-x^*|^2}{(1-L_0|x_0-x^*|)^2\beta|\alpha|} \\
&\quad + \frac{M(\beta+1)\varphi(|x_0-x^*|)|x_0-x^*|}{|\beta|(1-L_0|x_0-x^*|)} \\
&= g_2(|x_0-x^*|)|x_0-x^*| \le |x_0-x^*| < r,
\end{aligned} \tag{53}
$$

which shows (45) for $n = 0$ and $x_1 \in U(x^*, r)$.

The rest of the proof for estimates (44) and (45) follows using induction by simply replacing x_0, y_0, x_1 by x_n, y_n, x_{n+1} in the preceding estimates. Then, from the estimate

$$|x_{n+1} - x^*| \le c|x_n - x^*| \le |x_n - x^*| < r, \qquad c = g_2(|x_0 - x^*|) \in [0, 1),$$

we deduce that $\lim_{n\to\infty} x_n = x^*$ and $x_{n+1} \in U(x^*, r)$. Finally, to show the uniqueness part, let $Q = \int_0^1 f'(x^* + \theta(y^* - x^*))d\theta$ with $f(y^*) = 0$ and $y^* \in D_2$. In view of (39), we get that

$$
\begin{aligned}
|f'(x^*)^{-1}(Q - f'(x^*))| &\le L_0 \int_0^1 |y^* + \theta(x^* - y^*) - x^*| \\
&\le L_0 \int_0^1 (1-\theta)|(x^* - y^*)|d\theta = \frac{L_0}{2}q < 1.
\end{aligned} \tag{54}
$$

Hence, $Q \ne 0$. Then, in view of the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$.

Remark 0.1 *1. It follows from* (39) *that condition* (41) *can be dropped, if we set*

$$M(t) = 1 + L_0 t$$

or

$$M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

2. *The point* r_A *is the convergence radius of Newton's method*

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \text{ for each } n = 0, 1, 2, \dots \tag{55}$$

given by us in [2]. It follows from (33) *that the convergence radius* r *of method* (9) *is smaller than* r_A.

3. *Let us show how to choose function* φ, *when* H *is defined by*

$$H(x) = 1 - \frac{9(\beta - 1)}{32}\Big(\frac{f'(x - \alpha f'(x)^{-1} f(x))}{f'(x)}\Big). \tag{56}$$

In view of the proof of Theorem 2 and (56), *we have that*

$$\begin{aligned}|H(\tau) - 1| = \frac{9}{32}|\beta - 1|\Big|\frac{f'(x^*)^{-1} f'(y_n)}{f'(x^*)^{-1} f'(x_n)}\Big|^2 &\leq \frac{9}{32}|\beta - 1|\frac{M^2}{(1 - L_0|x_n - x^*|)^2} \\ &= \varphi(|x_n - x^*|).\end{aligned}$$

If we choose

$$\varphi(t) = \frac{9}{32}|\beta - 1|\Big(\frac{M}{1 - L_0 t}\Big)^2. \tag{57}$$

Next, we complete this section with some examples by choosing function φ as in (57).

Example 0.1 *Let* f *be a function defined on* $D = \bar{U}(0, 1)$, *which is given as follows*

$$f(x) = e^x - 1.$$

Then, $f'(x) = e^x$ *and* $x^* = 0$. *We get that* $L_0 = e - 1 < L = e^{\frac{1}{L_0}}$ *and* $M = e^{\frac{1}{L_0}}$. *The parameters using method* (9) *are:*

$$r_1 = 0.154407, \; r_p = 0.100312, \; r_2 = 0.138045, \; r_A = 0.382692$$

and as a consequence

$$r = 0.100312.$$

Example 0.2 *Returning back to the motivational example, we have that* $L = L_0 = 146.6629073$, $M = 2$ *and* $L_1 = L$. *The parameters using method* (9) *are:*

$$r_1 = 0.001515, \; r_p = 0.001138, \; r_2 = 0.004589, \; r_A = 0.00454557$$

and as a consequence

$$r = 0.001138.$$

Numerical experiments

In this section, we shall check the convergence behavior of newly proposed scheme (9) using weight functions (27), (29) and (31) (for $\beta = \frac{3}{4}$) to solve some nonlinear equations given in Table 4, which serve to check the validity and efficiency of theoretical results. These methods are denoted by $OM1$, $OM2$ and $OM3$, respectively. We compare them with existing robust methods, namely, Jarratt's method JM (3), Soleymani's methods (4) ($S1$), (5) ($S2$), Khattri and Abbasbandy method (6) (ABK), respectively. All computations have been performed using the programming package *Mathematica* 7 [5] in multiple precision arithmetic environment. We have considered 2000 digits floating point arithmetic so as to minimize the round-off errors as much as possible.

To check the theoretical order of convergence, we calculate the computational order of convergence *(COC)* [8] denoted by ρ_c using the following formula

$$\rho_c = \frac{\log\left(|f(x_n)/f(x_{n-1})|\right)}{\log\left(|f(x_{n-1})/f(x_{n-2})|\right)}, \; n = 2, 3, ...,$$

by taking into consideration the last three approximations in the iteration process. We have considered variety of test functions of different nature to compute the errors $|x_n - x^*|$ of approximations.

For instance, we consider the following Planck's radiation law problem which calculates the energy density within an isothermal blackbody and is given by [20]:

$$\Psi(\lambda) = \frac{8\pi c P \lambda^{-5}}{e^{\frac{cP}{\lambda BT}} - 1}, \tag{58}$$

where λ is the wavelength of the radiation, T is the absolute temperature of the blackbody, B is the Boltzmann constant, P is the Planck constant and c is the speed of light. We are interested in determining wavelength λ which corresponds to maximum energy density $\Psi(\lambda)$.

Further, $\Psi'(\lambda) = 0$ implies that the maximum value of Ψ occurs when

$$\frac{\frac{cP}{\lambda BT} e^{\frac{cP}{\lambda BT}}}{e^{\frac{cP}{\lambda BT}} - 1} = 5. \tag{59}$$

If $x = \frac{cP}{\lambda BT}$, then (59) is satisfied when

$$f_1(x) = e^{-x} + \frac{x}{5} - 1 = 0. \tag{60}$$

Therefore, the solutions of $f_1(x) = 0$ give the maximum wavelength of radiation λ by means of the following formula:

$$\lambda \approx \frac{cP}{x^* BT}, \tag{61}$$

where x^* is a solution of (60).

Now, let us consider the test function f_3 is a polynomial of *Wilkinson's type* with real zeros $1, 2, 3, 4, 5$. It is well-known that this class of polynomials is ill-conditioned and small perturbations in polynomial coefficients cause drastic variations of zeros. Therefore, most of the iterative methods encounter serious difficulties in finding the zeros of *Wilkinson-like polynomials*. The errors $|x_n - x^*|$ of approximations to the corresponding zeros of test functions and computational order of convergence ρ_c are displayed in Table 5, where

Table 4: Test functions and their zeros

$f(x)$	*Root* (x^*)	*Initial Guess* (x_0)
$f_1(x) = e^{-x} + \frac{x}{5} - 1$	4.965114	5.2
$f_2(x) = (x-1)^3 - 1$	2	2.8
$f_3(x) = \prod_{i=1}^{5}(x-i)$	4	4.5
$f_4(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$	-1	0.2
$f_5(x) = xe^{x^2} - \sin x^2 + 3\cos x + 5$	$-1.201576\ldots$	0.9
$f_6(x) = e^{-x^2+x+2} - 1$	-1	-0.6

Table 5: Comparison of different optimal fourth-order methods

$f(x)$		*JM*	*S*1	*S*2	*ABK*	*OM*1	*OM*2	*OM*3
f_1	$\lvert x_1 - x^*\rvert$	0.708e−6	0.301e−2	0.704e−3	0.800e−6	0.703e−6	0.701e−6	0.701−6
	$\lvert x_2 - x^*\rvert$	0.709e−28	0.821e−10	0.245e−15	0.136e−27	0.684e−28	0.675e−28	0.671e−28
	$\lvert x_3 - x^*\rvert$	0.107e−48	0.454e−40	0.107e−48	0.107e−48	0.104e−43	0.107e−48	0.107e−48
COC (ρ_c)		4.0000	4.0000	4.0427	4.0000	4.0000	4.0000	4.0000
f_2	$\lvert x_1 - x^*\rvert$	0.528e−1	0.176e−1	0.480e−1	0.550e−1	0.272e−2	0.222e−2	0.595e−3
	$\lvert x_2 - x^*\rvert$	0.459e−5	0.201e−6	0.109e−4	0.586e−4	0.141e−10	0.254e−12	0.117e−13
	$\lvert x_3 - x^*\rvert$	0.296e−21	0.384e−26	0.373e−19	0.107e−15	0.104e−43	0.433e−52	**0.178**e−56
COC (ρ_c)		3.9647	3.9855	3.9502	3.9176	3.9987	4.0002	3.9998
f_3	$\lvert x_1 - x^*\rvert$	0.108e+0	*CUR*	*CUR*	0.217e+0	0.107e+0	0.105e+0	0.106e+0
	$\lvert x_2 - x^*\rvert$	0.292e−3	*CUR*	*CUR*	0.393e+0	0.224e−3	0.139e−3	0.164e−3
	$\lvert x_3 - x^*\rvert$	0.851e−14	*CUR*	*CUR*	0.111e−1	0.238e−14	**0.297**e−15	**0.612**e−15
COC (ρ_c)		4.1747	—	—	3.8388	4.1652	4.1170	4.13333
f_4	$\lvert x_1 - x^*\rvert$	0.217e+0	*Div*	*Div*	0.219e+0	0.217e+0	0.217e+0	0.217e+0
	$\lvert x_2 - x^*\rvert$	0.176e−4	*Div*	*Div*	0.198e−4	0.176e−4	0.175e−4	0.175e−4
	$\lvert x_3 - x^*\rvert$	0.746e−20	*Div*	*Div*	0.586e−20	**0.752**e−20	**0.753**e−20	**0.753**e−20
COC (ρ_c)		3.7643	—	—	3.8464	3.7615	3.7598	3.7604
f_5	$\lvert x_1 - x^*\rvert$	0.266e−2	0.167e+0	0.119e+0	*Div*	0.661e−2	0.819e−2	0.450e−2
	$\lvert x_2 - x^*\rvert$	0.205e−10	0.468e−2	0.935e−3	*Div*	0.150e−8	0.651e−8	0.499e−9
	$\lvert x_3 - x^*\rvert$	0.475e−34	0.428e−8	0.451e−11	*Div*	**0.513**e−34	0.269e−32	**0.476**e−34
COC (ρ_c)		3.9994	3.6276	3.8074	—	4.0001	4.0022	4.0009
f_6	$\lvert x_1 - x^*\rvert$	0.102e−1	0.121e−1	0.160e−1	0.352e−1	0.423e−2	0.316e−2	0.354e−2
	$\lvert x_2 - x^*\rvert$	0.105e−7	0.846e−7	0.242e−6	0.171e−4	0.106e−9	0.585e−11	0.112e−10
	$\lvert x_3 - x^*\rvert$	0.120e−31	0.216e−27	0.138e−25	0.123e−17	0.417e−40	**0.710**e−46	0.112e−44
COC (ρ_c)		3.9950	3.9895	3.9859	3.9449	3.9976	3.9976	3.9989

CUR: Convergence to undesired root and *Div*: stands for Divergence

Note: Bold-face numbers denote the least error among the displayed methods.

$A(-h)$ denotes $A \times 10^{-h}$. On the accounts of results obtained in the Table 5, it can be concluded that the proposed methods are highly efficient as compared to the existing robust methods, when the accuracy is tested in the multi-precision digits. Additionally, the computational order of convergence *(COC)* of these methods also confirmed the above conclusions to a great extent.

Furthermore, we have also included two pathological examples to show that our proposed methods (9) will converge to the complex root without having to start with a complex number.

Example 1: $g_1(x) = x^3 - 3x^2 + 2x + \frac{2}{5}$.

In this pathological example, starting from the real initial guess $x_0 = 1.5$, our methods namely, $OM1,\ OM2,\ OM3$ for $(\beta = 3/4)$ takes only 5 iterations to converge to the complex root $1.57985 - 0.0932014I$ with error in the approximation as $2.4264e\text{-}6 - 4.39e\text{-}8I$. On the other hand, other existing methods fail to give complex roots starting from any real guess.

Example 2: $g_2(x) = x^3 + 2x^2 + 5$.

The zeros here are -2.69065, $0.345324 - 1.31873I$ and $0.345324 + 1.31873I$. Starting with real initial guess x_0 in our methods, we shall get a complex root. For instance, starting from the real initial guess $x_0 = 0.4$, our methods namely, $OM1,\ OM2,\ OM3$ for $(\beta = \frac{3}{4})$ takes only 6 iterations to converge to the complex root $0.345324 - 1.31873I$ with error in the approximation as $0.000e\text{-}17{+}2.6375I$. The other existing methods get no solution, no matter how many iterations are performed. This also demonstrates the advantage of our methods in finding complex roots without having to start with a complex initial guess.

Similar numerical experiments have been carried out on variety of problems which confirm the above conclusions to a great extent. Finally, we observe that our proposed methods have better stability and robustness as compared to the other existing methods.

Conclusions

In this study, we contribute further to the development of the theory of iteration processes and propose new fourth-order variants of Jarratt type methods for solving nonlinear equations numerically. The presented scheme is optimal in the sense of Kung-Traub conjecture and includes optimal modifications of Ostrowski's square root method, Euler's method and Laguerre's method for different values of free disposable parameter. Moreover, the local convergence of these methods is also given using hypotheses only on the first derivative and Lipschitz constants. The another most striking feature of this contribution is that the proposed methods can locate the complex roots without having to start from a complex number as would be necessary with other methods. Finally, the asserted superiority of the proposed methods is also corroborated in the numerical section.

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