A reduced-order modeling technique for nonlinear buckling analysis

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Abstract:

A reduced-order modeling technique, termed the Koiter-Newton method, is presented for the elastic nonlinear structural problems. It is a combination of Koiter analysis and Newton arc-length method so that it is accurate over the whole equilibrium path but is also efficient in the presence of buckling. Various numerical examples are presented to evaluate the performance of the method.

Keywords: Koiter-Newton approach, Nonlinear structural problems, Koiter analysis, Buckling

Introduction

Nonlinear static analysis of structures is an essential step of the design of flight vehicles. It is also important in many situations of practical interest. For example, it is crucial when the displacements and/or rotations are large. Even more importantly for flight vehicles is the case where the structure (or some of its components) are prone to buckling. In many cases, it is crucial to assess the loads at which buckling occurs as well as the behavior of the structure beyond the buckling point (usually termed post-buckling analysis) [1].

Traditionally, there have been two major approaches to this problem. The first approach is the reduction method which is based on the physics-based reduced order models[2][3]. The basic idea is to significantly reduce the number of degrees of freedom in a nonlinear finite element model. Several orders of magnitude reduction in model size is possible using this approach[4][5]. This method can be implemented in a finite element environment[2][4] and applied to moderately complex structures. Basically, there are two kinds of reduction methods. One is the Koiter's reduction method which is based on the Koiter's celebrated initial post-buckling theory [6]-[8]. It is very good for dealing with buckling sensitive structures and closely spaced modes. But it is based on just one perturbation expansion and is valid only in a small range around the buckling point. The other one is the general reduction method based on the power series expansion[9][10]. The expansions are carried out on some points along the equilibrium path in a step by step manner so that it can trace the whole nonlinear path. However, it is not good for dealing with the buckling sensitive structures. In addition, for both of the two reduction methods, there is no further link between the original finite element model and the reduced order model. Thus, the range of validity of the approximate model needs to be assessed by comparison to a full finite element analysis. This situation greatly limits the applicability of the new approach. The second approach is the finite element analysis. In this approach, the nonlinear response is traced along the equilibrium path starting from the nondeformed position by a traditional Newton method. Now, this approach finds difficulty in tracing the response of buckling sensitive structures especially if the structure has closely spaced buckling modes[11]. In addition, this method is usually very expensive for computing the large nonlinear equations.

To achieve greater applicability, a combination of Koiter analysis and Newton arc-length method is proposed in this paper. In this Koiter-Newton approach, a reduced order model (ROM) is constructed based on the Koiter's initial post-buckling theory. This ROM is used to make an initial prediction of the response of the structure. At the new predicted point, the exact unbalanced force residual is calculated using the full finite element model. Then in a corrector step, this residual is driven to zero similar to traditional Newton arc-length methods. As the solution proceeds to higher and higher load levels, the quality of the ROM are assessed (based on the norm of force residuals) and if needed the ROM is updated to reflect changes in structural stiffness and load distribution. The proposed approach will significantly improve the efficiency of nonlinear static finite element analysis by incorporating information from Koiter's analysis while retaining the complete generality usually associated with finite element modeling.

Koiter Newton Approach

The nonlinear equilibrium equations can be written as the following simple form. It is ended with the third order about the displacement u,

$$L(u)+Q(u,u)+C(u,u,u)=f\cdot\lambda=F \tag{1}$$

where L is a linear operator, Q is a quadratic one and C is a cubic one. f is a matrix whose columns are formed by the sub-loads f_p. λ is the load parameter vector. The multiple load F is a summation of the sub-loads multiplied by the corresponding load parameters,

$$F=\sum_{p=1}^{m+1}\lambda_p f_p \tag{2}$$

where, m+1 is the total number of degrees of freedom in the reduction method. m is the number of degrees of freedom which is used in the analysis for describing the buckling branches and it is associated with the number of the closed buckling modes of the structure. 1 is the general degree of freedom for the primary path.

The displacement is also expanded to the third order with respect to the perturbation parameter a,

$$u=a_i u_i+a_i a_j u_{ij}+a_i a_j a_k u_{ijk} \tag{3}$$

where, the subscripts i,j,k=1,2,...,m+1. In the first order displacements u_i, u_1 is the displacement in the primary path; u_i(i=others) is the buckling branches. The second order displacements u_{ij} and third order displacements u_{ijk} describe the interaction effect of different first order displacement fields.

The finial reduced order model is assumed to be,

$$\lambda = \bar{L}(a) + \bar{Q}(a,a) + \bar{C}(a,a,a) \tag{4}$$

where, the $\bar{L}$, $\bar{Q}$ and $\bar{C}$ are separately the linear, quadratic and cubic operator. Introducing the equation (3) and (4) to the both sides of the equilibrium equation (1) and equating the coefficients of the various powers of a to zero, it will yield three linear equations.

$$\begin{bmatrix} K_t & -f \\ -f^T & 0 \end{bmatrix} \begin{Bmatrix} u_i \\ \bar{L}_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ -E_i \end{Bmatrix} \tag{5}$$

$$\begin{bmatrix} K_t & -f \\ -f^T & 0 \end{bmatrix} \begin{Bmatrix} u_{ij} \\ \bar{Q}_{ij} \end{Bmatrix} = \begin{Bmatrix} -Q(u_i, u_j) \\ 0 \end{Bmatrix} \tag{6}$$

$$\bar{C}_{pijk} = C(u_i, u_j, u_k, u_p) - \frac{2}{3}\left(u^t_{ij} \cdot L(u_{pk}) + u^t_{jk} \cdot L(u_{pi}) + u^t_{ki} \cdot L(u_{pj}) \right) \tag{7}$$

where, the subscripts i,j,k,p=1,2,...,m+1. $K_t = \bar{L}$ is the tangent stiffness matrix. In the vector E_i, only the i_{th} component is equal to one and the other components are all equal to zero. It is easy to see that only the first two linear equations need the matrix triangulation and they two have the same coefficient matrix. After solving them, the $\bar{L}$, $\bar{Q}$, $\bar{C}$, u_i, u_{ij} can be obtained. Then, the specific expression of the ROM is generated. Using the arc-length method to solve the ROM, the relationship for the load parameter λ and perturbation parameter a will be known. Introducing this relationship into the expansion of the displacement (3), the nonlinear response of the structure(λ - u) will be obtained.

In order to have an efficient algorithm, the analysis of the range of validity and the definition of a new starting point should be automatic, i.e. we have to automatically determine the values of the displacement u, over which the reduced solution will not satisfy a given accuracy.

In each step (or expansion) of the Koiter-Newton approach, this ROM is used to make an initial prediction of the response of the structure. During solving the ROM, the exact unbalanced force is calculated using the full finite element model at the end of each solution step. A criterion about the unbalanced force is given to judge when the initial prediction should be ended. If the criterion is not satisfied, the initial prediction will be stopped. Then in the following corrector step, this residual RF will be driven to zero similar to traditional Newton arc-length methods. The convergent point on the equilibrium path will be a new starting point for the next expansion. Until now, one whole step for the Koiter-Newton approach is ended. The path-tracing strategy of the proposed method is illustrated in Fig. 1. The proposed method has a larger step size to trace the nonlinear equilibrium path of the structure, compared to the conventional Newton method. This makes the method be a computationally efficient technique.

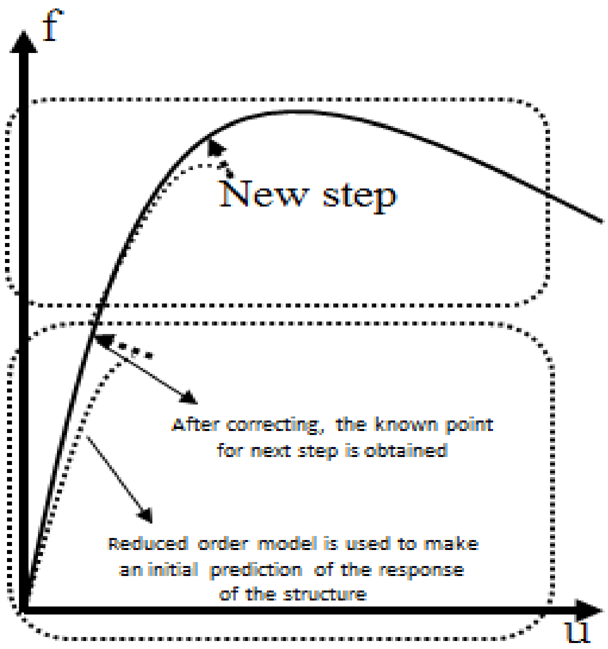
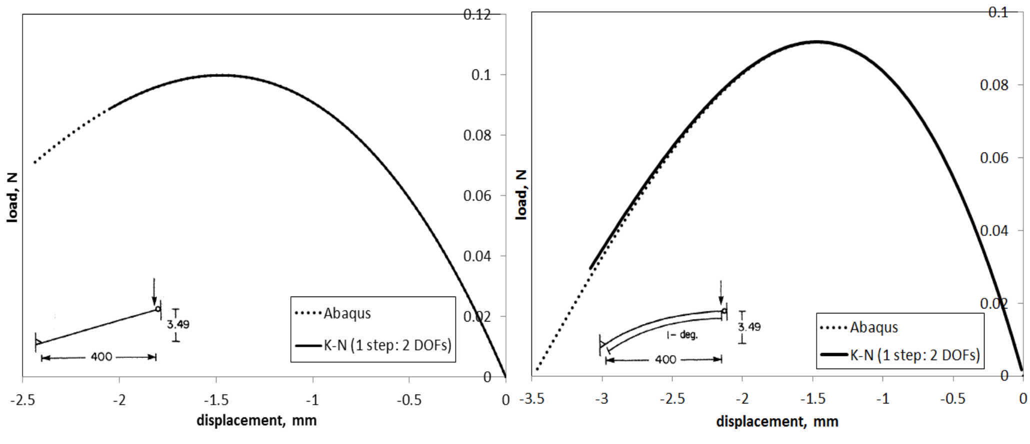
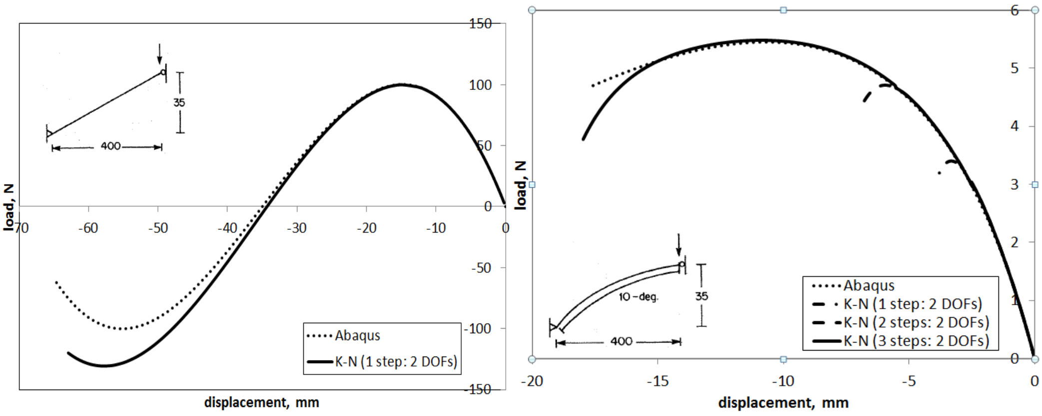


Figure 1. Path-following strategy of the proposed method

Numerical Examples

Six beams[12]which all have a nonlinear prebuckling state are analyzed in this example. They have the different shape, depth, constrain condition and loading position, as showed in figure 2. Young's modulus are all 2000MPa. The area and moment of inertia of the cross section are 391mm^2 and 2000mm^4, respectively.





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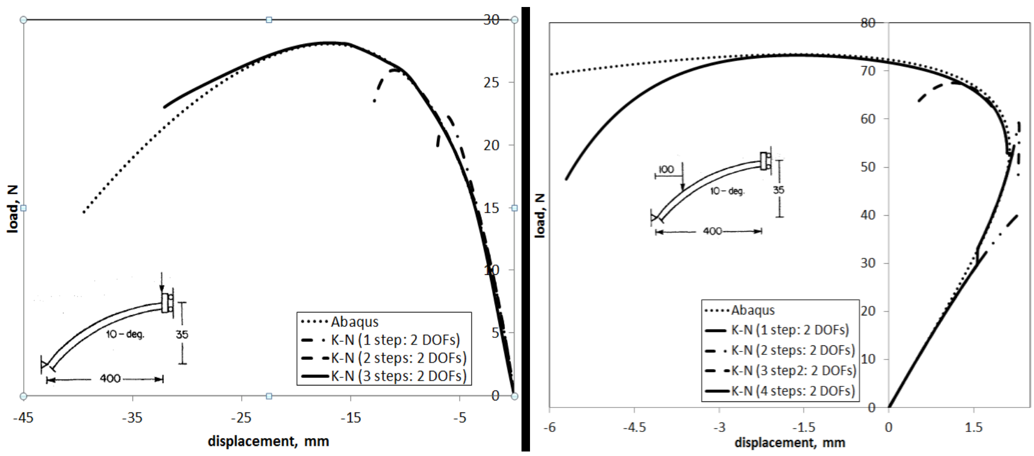

Figure 2 Buckling response curves for the six single beams

Koiter-Newton approach is used to analyze these six beams. Because the first buckling load are not closed with the others, only the first buckling mode will be chosen and the number of degrees of freedom in the reduced order model is two. The nonlinear response curves (vertical displacement on the loading point vs. loading) compared with the Abaqus are in figure 2. For the beams(a)~(c), the figures show that only one perturbation step is enough to obtain an accurate buckling response(including prebuckling state, limit load and initial postbuckling state). However, because of the extremely nonlinearity of beams(e)~(g), 3, 3 and 4 steps will be needed to follow the nonlinear buckling paths, separately.

The computing consumption for reaching the same point on the postbuckling path is compared with the Abaqus. Here, the numbers of the linear equations needed to be solved are listed on table 1 for comparison. It is obviously that the Koiter-Newton approach is much more efficient than Abaqus.

Table 1. Comparison of the computing time

Beam examples	(a)	(b)	(c)	(d)	(e)	(f)
Abaqus	6	9	10	28	39	56
Koiter-Newton method	1	1	1	9	12	12

Conclusions

Based on the Koiter's initial post-buckling theory and the incremental iterative technique of the Newton method, a new reduction method, that is the Koiter-Newton method, which can trace the whole nonlinear equilibrium path automatically, is proposed. Co-rotational elements are successfully implemented into this new method. Some classical numerical examples are used to evaluate the Koiter-Newton method. If prebuckling nonlinearity is not very serious, only one perturbation step is enough to obtain the buckling characteristic. Otherwise, more steps will be needed due to the serious nonlinearity of the prebuckling. By comparing the results with Abaqus which adopts the full nonlinear finite element method, it proves that the Koiter-Newton method is automatically, accuracy and more efficient.

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