Interval field model and interval finite element analysis

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Abstract

Uncertain parameters with inherent spatial variability are commonly encountered in engineering. Modeling of this kind of spatial uncertainty plays a fundamental role in structural uncertainty analysis, which provides a necessary basis for subsequent uncertainty propagation through the system. An interval field model for quantification of spatial uncertain parameters is proposed, by which only the upper and lower bounds of the spatial uncertain parameters rather than their precise probability distributions are required. The dependency can be fully considered by the proposed interval field model. With the information of dependency, an interval K-L expansion is presented as a combination of deterministic functions with uncorrelated standard interval variables, through which the continuous spatial interval field with dependency can be expressed only by very limited intervals. Necessary mathematical illustrations are provided for the proposed interval field model and the interval K-L expansion. The sampling method for the interval field model is given, providing a robust numerical analysis basis for subsequent structural uncertainty analysis. When the interval field model is applied in finite element analysis of structures with spatial uncertain parameters, the non-deterministic equilibrium equations with interval factors is then formulated. The MCS method and the perturbation method are developed for solution of the derived interval equilibrium equations. The feasibility and validity of the proposed interval field model and corresponding interval finite element methods are verified by numerical examples, where the upper and lower bounds of the responses such as the displacements and the stresses of structures with spatial uncertain parameters are computed and compared.

Keywords: Interval field model; Spatial uncertainty; Interval K-L expansion; Interval finite element method

1. Introduction

The modeling of uncertain input parameters with inherent spatial variability are commonly encountered in engineering. These include material properties of the heterogeneous media such as concrete or porous rock, geographical parameters such as soil permeability over the scale of meters. This kind of uncertainties generally present with spatially varying properties, which traditionally can be well quantified by random field models [1]. And solutions of the stochastic problems where the properties of the structures are modeled as random fields can be found by stochastic finite element method (SFEM) [2]. As a primary non-deterministic framework, the probabilistic methods have been tremendously developed over the last decades. However, a large amount of information is required to determine the credible probability density function (PDF) for construction of the probabilistic model, which is quite impractical or very costly to obtain in many engineering problems.

In this work, an interval field model is proposed for quantification of a spatially uncertain parameter, which requires only the upper and lower bounds of the parameter rather than its precise probability distributions. An interval field can be denoted as $\left\{H(\mathbf{x})\in H^I(\mathbf{x}),\mathbf{x}\in D\right\}$, where D refers to spatial domain. In Fig. 1, an interval field with constant upper and lower bounds is given. For arbitrary location $\mathbf{x}_k$ in the two-dimensional domain, the variation range of the variable $H(\mathbf{x}_k)$ is strictly limited within the interval $H^I(\mathbf{x}_k)=[H^L(\mathbf{x}_k),H^U(\mathbf{x}_k)]$. In many practical circumstances, although the values of a spatial uncertain parameter differ with location, dependency exists between these spatial uncertainties especially for those adjacent ones. For example, the material property such as the elasticity modulus of a concrete structure may be spatially uncertain because of its inhomogeneity, however the value of the elasticity modulus at arbitrary location is likely to be close to that at another location nearby. For this reason, the covariance function $C(\mathbf{x},\mathbf{x}')$ and the correlation coefficient function $R(\mathbf{x},\mathbf{x}')$, $\mathbf{x},\mathbf{x}'\in D$ are also defined to reflect the dependency degree of the interval field at different locations.

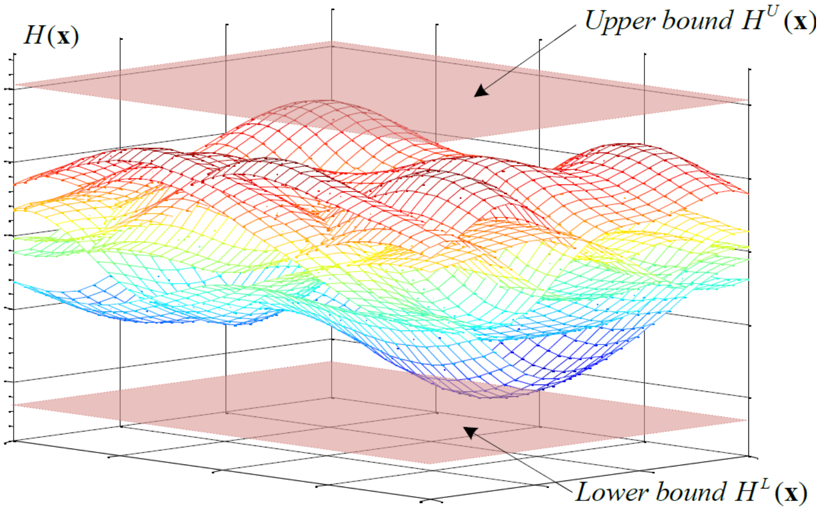

Fig.1. Interval field model

Similar to the Karhunen-Loève (K-L) expansion of a random field [2], the interval K-L expansion is created to represent an interval field as an infinite linear combination of orthogonal functions multiplied with uncorrelated standard interval variables, which can be expressed as:

$$H(\mathbf{x})=H^c(\mathbf{x})+\sum_{j=1}^{\infty}H^r(\mathbf{x})\sqrt{\lambda_j}\varphi_j(\mathbf{x})\zeta_j \qquad (1)$$

where $H^c(\mathbf{x})$ and $H^r(\mathbf{x})$ are the midpoint function and the radius function of the interval field, respectively; $\zeta_j\in\zeta^I=[-1,1]$, $j=1,2,\ldots$ are standard uncorrelated interval variables that satisfy $\sum_{j=1}^{\infty}\zeta_j^2\le 1$; and $\lambda_j\in[0,\infty)$, $\varphi_j(\mathbf{x}):D\to\mathbb{R}$ are respectively the eigenvalues and eigenfunctions of the correlation coefficient function $R(\mathbf{x},\mathbf{x}')$. Corresponding mathematical foundations for this interval field model and the interval K-L expansion are also established. In practical engineering, it is not only impossible but also unnecessary to use infinite terms with interval variables for quantification of the spatial uncertainty. Generally, most of the characteristics of a spatially uncertain parameter can be reflected considerably by those principle terms. Therefore for practical implementation, the series is generally approximated by sorting the eigenvalues λ_i and the corresponding eigenfunctions $\varphi_j(\mathbf{x})$ in a descending order and truncating the expansion after M terms. The error analysis of the truncated form in the representation of a spatially uncertain parameter is also given, from which an index that evaluates the degree of approximation is suggested.

When the interval field model is applied to the finite element analysis of structures with spatial uncertain parameters such as material properties and distributed loads, the derivation of the interval finite element method (IFEM) [3, 4] is then formulated. According to the sources of spatial uncertainty $\alpha(\mathbf{x})$ in the finite element system, the non-deterministic equilibrium equation can be classified into the following three types:

$$\begin{array}{ll} \text{Type-I} & \mathbf{K}(\alpha)\mathbf{u}=\mathbf{p} \\ \text{Type-II} & \mathbf{K}\mathbf{u}=\mathbf{p}(\alpha) \\ \text{Type-III} & \mathbf{K}(\alpha)\mathbf{u}=\mathbf{p}(\alpha) \end{array} \tag{2}$$

For the existence of the spatially uncertain parameters with variation bounds, the responses such as the displacements and stresses of a structure also presents with bounded uncertainties. Both the Monte Carlo simulation (MCS) method and the perturbation method are developed to solve the interval equilibrium equations derived by the IFEM. The procedure of the MCS method is given in Fig. 2, which provides a robust numerical analysis framework and can be used as a standard reference for other numerical solutions.

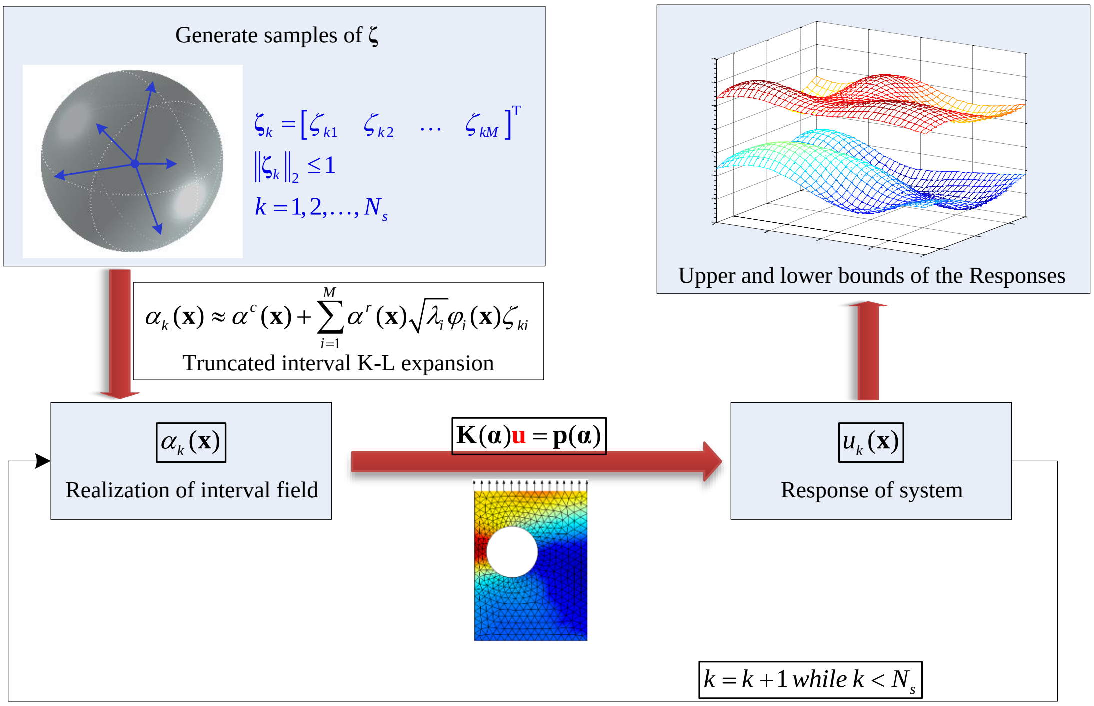

Fig. 2. MCS method for evaluation of response bounds

2. Results and discussions

Displacement analysis of a concrete quadrate plate subjected to distributed forces at the two sides is implemented. Due to the inhomogeneity and the spatial uncertainty of the material properties of the concrete plate, the Young's modulus is described as an interval field $E(\mathbf{x}) \in E^I(\mathbf{x})$ with constant midpoint function $E^c(\mathbf{x}) = 32.5\text{GPa}$ and radius function $E^r(\mathbf{x}) = 10\% E^c(\mathbf{x})$. The correlation coefficient function $R(\mathbf{x}, \mathbf{x}')$ is given as exponential form. With the truncated interval K-L expansion, the continuous spatial uncertain Young's modulus $E(\mathbf{x})$ is approximated only by 24 standard uncorrelated interval variables with approximation degree $\kappa = 90.42\%$. The response bounds of horizontal displacements by the perturbation method are depicted in Fig. 3. The region enveloped by the upper bound and the lower bound indicates the variation domain of all possible responses under all realizations of the spatially uncertain Young's modulus $E(\mathbf{x})$. In general, the perturbation method can be regarded as an effective approach for problems with uncertainty of degree not higher than 10%. For problems with large uncertainty, the MCS method can be applied, and other more effective methods are required to be developed in future.

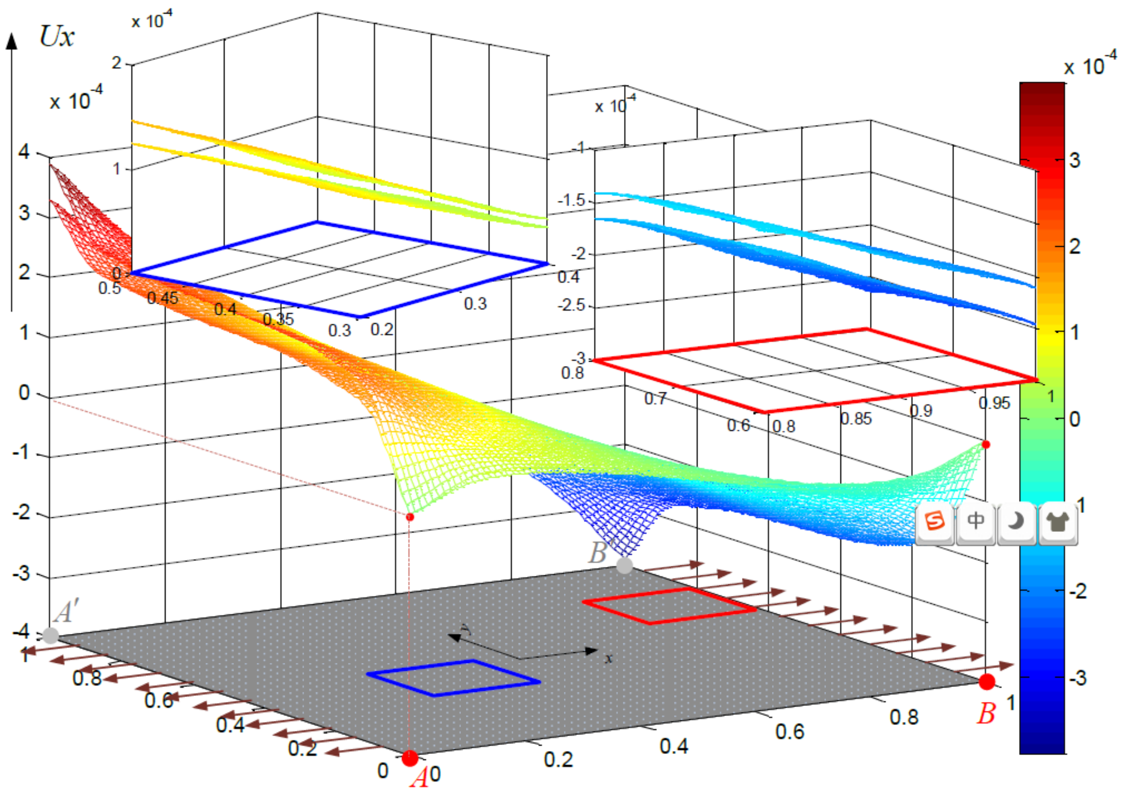

Fig. 3. Response bounds of horizontal displacement by the perturbation method

3. Conclusions

In this work, an interval field model for quantification of spatially uncertain parameters is proposed. The interval field model requires only the upper and lower bounds of the spatial uncertain parameters rather than their precise probability distributions. The dependency can be fully considered by the proposed interval field model. With the information of dependency, an interval K-L expansion is presented as a combination of deterministic functions with uncorrelated standard interval variables, through which the continuous spatial interval field with dependency can be expressed only by very limited intervals. When the interval field model is applied in finite element analysis of structures with spatial uncertain parameters, the non-deterministic equilibrium equations with interval factors is then formulated. Solutions by MCS method and the perturbation method are developed and compared in numerical examples. The MCS method is applicable to cases of large uncertainties and strong nonlinearities, but it generally costs much computational time. The perturbation method is developed based on the assumption of small uncertainty of spatial parameters; it is an efficient approximation method for structural response analysis. In numerical examples, the upper and lower bounds of structural responses such as displacements and stresses are computed, by which the feasibility and validity of the proposed interval field model and corresponding interval finite element analysis method are illustrated.

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