An edge-based smoothed finite element method (ES-FEM) for acoustic problems using four-node quadrilateral elements

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Abstract

It is well known that standard finite element method (FEM) is unreliable to simulate the acoustic propagation problems addressed by the Helmholtz equation for high wavenumbers due to the 'numerical dispersion error'. This dispersion error is essentially caused by the 'overly-stiff' FEM model. In order to depress the dispersion error, an edge-based smoothed finite method (ES-FEM) is proposed to solve the acoustic problems using the four-node quadrilateral elements. In the ES-FEM model, the gradient field of the problem is smoothed using gradient smoothing operations over the smoothing domain. Owing to this edge-based gradient smoothing operation, the ES-FEM model behaves much softer the standard FEM model and hence can reduce the numerical dispersion error significantly in solving the acoustic problems. Numerical examples have been studied and the results verify the excellent properties of the present ES-FEM.

Keywords: edge-based smoothed finite element method (ES-FEM), gradient smoothing technique, acoustics, numerical dispersion error.

Introduction

The acoustic problems in an inviscid medium, governed by the Helmholtz equation play an important role in engineering practices. In recent years, the standard finite element method (FEM) are the most well-developed and widely-used numerical methods for solving the acoustic wave propagation problems [1-4]. Unfortunately, it is known that the FEM can only provide acceptable results in a lower frequency range because the so-called 'numerical dispersion error' caused by the 'overly-stiff 'nature of the FEM model can grow significantly with increasing the frequency range.

Recently, many improved numerical methods based on the standard finite element method have been proposed to reduce the numerical dispersion error for high frequency range, such as the Galerkin/least-squares finite element method (GLS) [5], the quasi-stabilized finite element method (QSFEM) [6], the residual-free finite element method (RFEM) [7] and the residual-based FEM [8] and so on. However, all of the above methods cannot successfully eliminate the numerical dispersion error in the high frequency range.

More recently, a series of smoothed finite element (S-FEM), named as node-based S-FEM (NS-FEM), edge-based S-FEM (ES-FEM) and face-based S-FEM (FS-FEM) was proposed by Liu et al [9]-[19]. In the S-FEM above, the ES-FEM has a ‘close-to-exact’ stiffness because the edge-based strain smoothing operation can provide a ‘proper softening effects’ to the FEM model. This neither ‘overly-stiff’ nor ‘overly-soft’ ES-FEM model has been successfully applied to the vibration and acoustic analysis.

In this work, the ES-FEM using quadrilateral mesh has been introduced to solve the 2D acoustic problems. The system stiffness matrix of the ES-FEM model is obtained by numerical integration over each smoothing domain associated with the edge of the quadrilateral. Numerical results demonstrates that the ES-FEM can provide more accurate solutions than the standard FEM with the same mesh., especially in the high frequency range.

Strong formulation of the acoustic problem

Consider the fluid inside a domain Ω with boundary Γ, assuming that the wave is a small harmonic perturbation of pressure around a steady uniform state and homogeneous, inviscid, compressible acoustic fluid can only undergo small translational movement. The second-order Helmholtz equation is given by:

$$\Delta p + k^2 p = 0 \text{, in } \Omega \tag{1}$$

where $k = \omega/c$ is the wave number, ω is the angular frequencies, p denotes the spatial distribution of the acoustic pressure, and c stand for the speed of sound traveling in the fluid.

In the acoustic analysis, the particle velocity v is linked to the gradient of the acoustic pressure through the equation of motion which can be written as:

$$\nabla p + j\rho c\omega v = 0 \tag{2}$$

where ∇ is the gradient operator and ρ represent the density of medium.

FEM and ES-FEM for acoustic analysis

Formulation of the FEM

The Galerkin weak form for the acoustic problems can be derived easily using the method of weighted residuals. By multiplying the strong form Eq. (1) with a weight or test function w in the entire domain, the weighted residual equation can be written as:

$$\int_\Omega w(\Delta p + k^2 p) d\Omega = 0 \tag{3}$$

Integrating by part and using Green's theorem, Eq. (3) can be further rewritten as:

$$-\int_{\Omega} \nabla w \cdot \nabla p d\Omega + k^2 \int_{\Omega} w \cdot p d\Omega - j\rho\omega \int_{\Gamma_N} w \cdot v_n d\Gamma - j\rho\omega A_n \int_{\Gamma_R} w \cdot p d\Gamma = 0 \quad (4)$$

In the above weighted residual form, the acoustic pressure can be expressed in the approximate form:

$$p = \sum_{i=1}^{m} N_i p_i = \mathbf{Np} \quad (5)$$

where m stands for the number of nodal variables of the element, N_i is the generalized FEM shape functions and p_i is the unknown nodal pressure.

In this work, the FEM shape function N is chosen as the weight function, so the weak form for acoustic problem can be expressed as:

$$-\int_{\Omega} (\nabla \mathbf{N})^{\mathbf{T}} \nabla \mathbf{NP} d\Omega + k^2 \int_{\Omega} \mathbf{N}^{\mathbf{T}} \mathbf{NP} d\Omega - j\rho\omega \int_{\Gamma_N} \mathbf{N}^{\mathbf{T}} v_n d\Gamma - j\rho\omega A_n \int_{\Gamma_R} \mathbf{N}^{\mathbf{T}} \mathbf{NP} d\Gamma = 0 \quad (6)$$

This is the so-called Galerkin weak form for acoustic problems and it can be finally expressed in the following standard matrix form:

$$[\mathbf{K} - k^2 \mathbf{M} + j\rho\omega \mathbf{C}]\{\mathbf{P}\} = -j\rho\omega \{\mathbf{F}\} \quad (7)$$

where $\mathbf{K}$ is the acoustical stiffness matrix, $\mathbf{C}$ is the acoustical damping matrix modeling Robin boundary condition, $\mathbf{M}$ is the acoustical mass matrix, $\mathbf{F}$ is the vector of nodal acoustical forces, and $\mathbf{P}$ is the nodal acoustical pressure in the domain:

$$\mathbf{K} = \int_{\Omega} (\nabla \mathbf{N})^{\mathrm{T}} \nabla \mathbf{N} \, \mathrm{d}\Gamma \quad (8)$$

$$\mathbf{C} = \int_{\Gamma_R} \mathbf{N}^{\mathrm{T}} \mathbf{N} A_n \, \mathrm{d}\Gamma \quad (9)$$

$$\mathbf{M} = \int_{\Omega} \mathbf{N}^{\mathrm{T}} \mathbf{N} \, \mathrm{d}\Omega \quad (10)$$

$$\mathbf{F} = \int_{\Gamma_N} \mathbf{N}^{\mathrm{T}} v_n \, \mathrm{d}\Gamma \quad (11)$$

$$\mathbf{P} = \{p_1, p_2, p_3, \cdots, p_n\} \quad (12)$$

Edge-based gradient smoothing operation

In this section, the gradient smoothing operation for 2D problems using four-node rectangular elements is formulated. In order to carry out the gradient smoothing operation, a background mesh of quadrilaterals is generated first. The element mesh shall have a total of N_n nodes and N_{eg} edges in the entire problem domain. Afterwards, the problem domain Ω is divided into N_s non-overlapping smoothing domains associated with the edges of the quadrilaterals, such that $\Omega=\sum_{k=1}^{N_s}\Omega_k^s$ and $\Omega_i^s \cap \Omega_j^s \neq \varnothing$, $i \neq j$, where N_s is the number of the smoothing domains. In this case, the number of the smoothing domains is equal the number of edges in the mesh: $N_s = N_{eg}$, as shown in Figure 1.

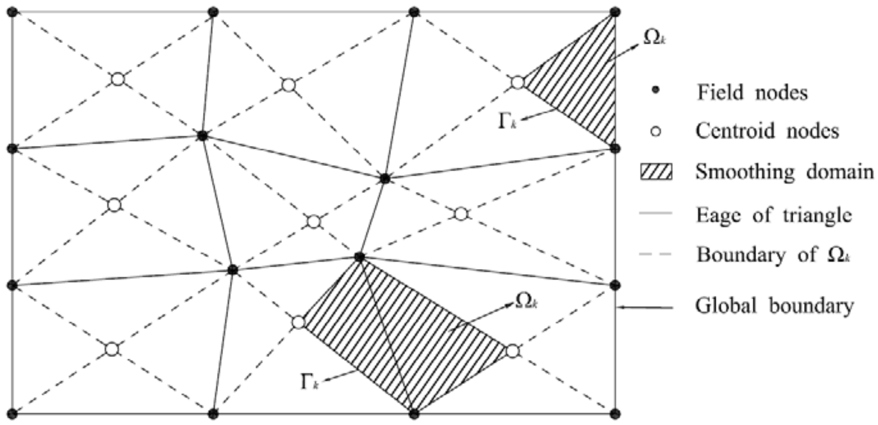


Figure 1. Edge-based smoothing domains in 2D problem

The acoustic particle velocity v is chosen as the primary field variable. The smoothed velocity can be expressed as [15]:

$$\bar{v}(\mathbf{x}_k)=\frac{1}{A_k}\int_{\Omega_k} v(\mathbf{x}_k)\,\mathrm{d}\Omega=-\frac{1}{j\rho\omega A_k}\int_{\Omega_k}\nabla p d\Omega=-\frac{1}{j\rho\omega A_k}\int_{\Gamma_k} p\cdot n d\Gamma \tag{13}$$

where $A_k=\int_{\Omega_k}\mathrm{d}\Omega$ is the area of the smoothing domain for edge k .

Using the FEM shape function in the Eq. (5), the smoothed velocity (or the smoothed pressure gradient) for edge k can be obtained as:

$$\bar{v}(\mathbf{x}_k)=-\frac{1}{j\rho\omega}\sum_{i\in M_k}\bar{\mathbf{B}}_i(\mathbf{x}_k)p_i \tag{15}$$

where $\bar{\mathbf{B}}_i$ is the smoothed gradient-pressure matrix and M_k is the total number of nodes in the influence domain of edge k . For two-dimensional problems:

$$\bar{\mathbf{B}}_i^T(\mathbf{x}_k)=\begin{bmatrix}\bar{b}_{i1} & \bar{b}_{i2}\end{bmatrix} \tag{16}$$

$$\bar{b}_{ip}=\frac{1}{A_k}\int_{\Gamma_k}\mathbf{N}_i(\mathbf{x})n_p(\mathbf{x})d\Gamma,\ (p=1,\ 2) \tag{17}$$

where n_p is the outward normal along the smoothing domain Ω_k^s. Using Gauss integration along each segment of boundary Γ_s^k, the above equation can be rewritten as:

$$\tilde{b}_{ip} = \frac{1}{A_s^k} \sum_{q=1}^{N_s} \left[\sum_{r=1}^{N_g} w_r N_i(\mathbf{x}_{qr}) n_p(\mathbf{x}_q) \right] \tag{18}$$

where N_g is the number of Gauss points distributed in each segment and w_r is the corresponding weight for Gauss point.

Formulation of the ES-FEM

The gradient component $\nabla\mathbf{N}$ in Eq. (6) is replaced by the smoothed item $\overline{\nabla\mathbf{N}}$ by introducing the edge-based gradient smoothing operation, then the smoothed Galerkin weak form for acoustic problems can be expressed as [15]:

$$-\int_{\Omega} (\overline{\nabla\mathbf{N}})^{\mathbf{T}} \overline{\nabla\mathbf{N}} \mathbf{P} d\Omega + k^2 \int_{\Omega} \mathbf{N}^{\mathbf{T}} \mathbf{N} \mathbf{P} d\Omega - j\rho\omega \int_{\Gamma_N} \mathbf{N}^{\mathbf{T}} v_n d\Gamma - j\rho\omega A_n \int_{\Gamma_R} \mathbf{N}^{\mathbf{T}} \mathbf{N} \mathbf{P} d\Gamma = \mathbf{0} \tag{19}$$

The smoothed discretized system equation can be obtained as:

$$[\overline{\mathbf{K}} - k^2 \mathbf{M} + j\rho\omega \mathbf{C}]\{\mathbf{P}\} = -j\rho\omega\{\mathbf{F}\} \tag{20}$$

which is the set of discretized equation for the ES-FEM models. The global ‘smoothed’ acoustic stiffness matrix $\overline{\mathbf{K}}$ is calculated as:

$$\overline{\mathbf{K}} = \int_{\Omega} (\overline{\nabla\mathbf{N}})^{\mathbf{T}} \overline{\nabla\mathbf{N}} d\Omega = \sum_{k=1}^{N_s} \overline{\mathbf{K}}_s^{(k)} \tag{21}$$

where $\overline{\mathbf{K}}_s^{(k)}$ is the smoothed element stiffness matrix for the smoothing domain Ω_k^s and it can be calculated as:

$$\overline{\mathbf{K}}_s^{(k)} = \int_{\Omega_s^k} (\overline{\nabla\mathbf{N}})^{\mathbf{T}} \overline{\nabla\mathbf{N}} d\Omega = A_s^k \overline{\mathbf{B}}^T \overline{\mathbf{B}} \tag{22}$$

where A_s^k is the area of the smoothing domain for edge k.

Numerical results

L-shaped cavity filled with air with Dirichlet boundary condition

Considering the 2D problem in an L-shaped acoustic domain as defined Figure 2. The acoustic domain with length L =1 m is filled with air (ρ =1.225 kg/m3, c =340 m/s). Dirichlet boundary condition with p =0.1 Pa are defined on the top boundary (y =1 m) and all other side is rigid with normal velocity v_n =0 m/s. This L-shaped acoustic domain is discretized into 100 nodes and 100 elements with average mesh size of 0.05 m.

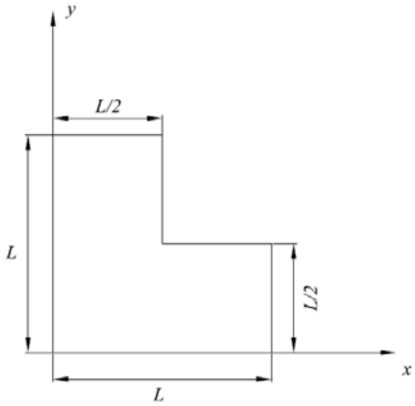


Figure 2. An L-shaped acoustic domain

Acoustic eigenfrequencies analysis

The acoustic eigenfrequencies analysis for this L-shaped domain using ES-FEM and FEM is conducted first. Table 1 lists the first ten non-rigid natural eigenfrequencies obtained from ES-FEM and FEM with the same mesh. In order to compare the two methods, the results computed using FEM with a highly refined mesh (300 nodes and 341 elements) is provided as a reference. As listed in Table 1, the ES-FEM is more accurate than the FEM among all the mode orders. It is indicated that in the ES-FEM model, the ‘overly-stiffness’ has been effectively softened owing to the edge-based gradient smoothing operation.

Table 1 The natural eigenfrequencies（Hz）of the L-shaped acoustic domain

Modes	Reference	FEM	Error of FEM	ES-FEM	Error of ES-FEM
1	131.53	132.07	0.41	131.69	0.12
2	203.45	203.69	0.11	203.47	0.01
3	340.01	341.06	0.31	340.11	0.03
4	340.01	341.18	0.34	340.11	0.03
5	365.25	366.64	0.38	365.36	0.03
6	383.82	385.98	0.56	384.10	0.07
7	480.84	484.20	0.70	480.96	0.02
8	501.08	505.73	0.93	501.65	0.11
9	522.92	526.94	0.77	523.21	0.05
10	577.81	584.44	1.15	578.34	0.09

Frequency response analysis

The frequencies response analysis for the L-shaped acoustic domain is then investigated using the ES-FEM and FEM with frequencies ranging from 1 to 1000 Hz. The boundary conditions and the quadrilateral meshes are the same as described previously. The frequencies response curves at point A (1.0m, 0.5m) obtained using the ES-FEM and the FEM are both plotted in Figure 3. As the analytical solution is unavailable, the reference solution using FEM with a very fine mesh (1976 nodes and 1875 elements) is also provided for comparison. Figure 3 illustrates the conclusions: the FEM can only provide very accurate results in low frequency range and the solution will become inaccurate with the increase of the frequency. While the ES-FEM can produce much more accurate results compared to the FEM, especially in high frequency range.

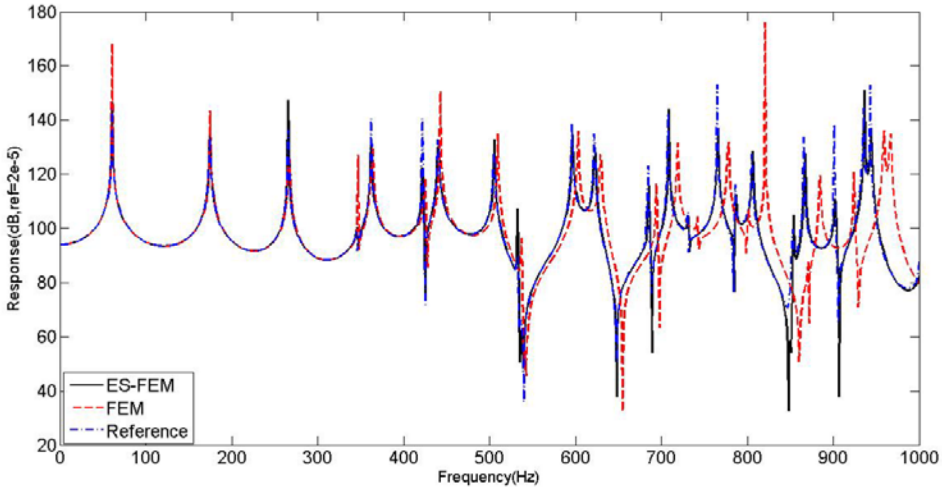
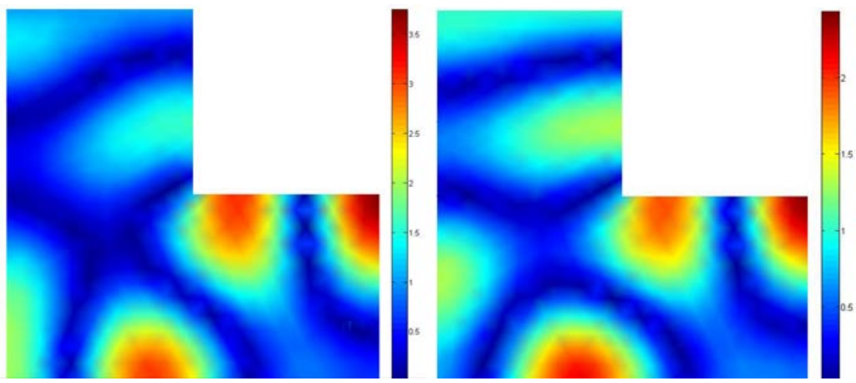
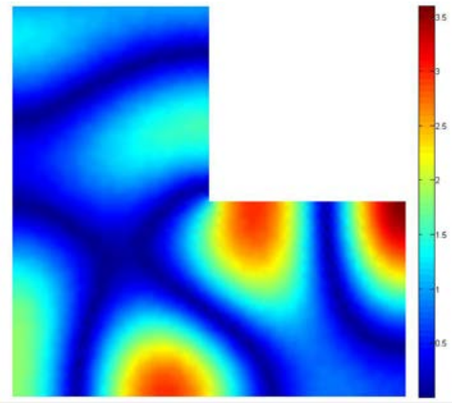


Figure 3. The frequency response at point A obtained using ES-FEM and FEM

In order to assess the performance of the ES-FEM further, the acoustic pressure distribution of the ES-FEM versus FEM with a frequency of 500 Hz are plotted in Figure 4a and Figure 4b, respectively. The reference result is also plotted in Figure 4c. As shown in these figures, the ES-FEM result are closer to the reference result than the FEM result. This also verifies the 'right-stiffness' of the ES-FEM for 2D acoustic problems.



(a) **(b)**



(c)

Figure 6. The acoustic pressure distribution of L-shaped domain: (a) ES-FEM (b) FEM (c) Reference (frequency=500Hz)

Conclusions

In this work, the ES-FEM using the four-node quadrilateral elements is formulated for the 2D acoustic problems. Through the numerical results, the following remarks can be made:

(1) The ES-FEM using quadrilateral elements in 2D problems is very simple and it can works well with fewer number of field nodes compared with the four-node isoparametric finite elements.

(2) Owing to the ‘overly-stiff’ feature of the FEM model, the FEM is sensitive to the frequency and can only achieve accurate result for low frequency range while the ES-FEM possesses a ‘close-to-exact’ stiffness due to the edge-based gradient smoothing technique. So the ES-FEM can provide higher accurate results than the FEM especially for high frequency range in acoustic analysis.

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