Computational method for geometric properties of arbitrary plane areas

†X. L. Chen[1], D. Y. Li[1], and L. Zhang[1]

[1]School of Mathematics and Physics, Chongqing University of Science and Technology, Chongqing, China.
†Corresponding author: xlchen80@163.com

Abstract

The centroid, the moment of inertia, and the product of inertia for an arbitrary polygon and an arbitrary plane area, are formulated in algebraic forms and programmed with Matlab software. Three numerical examples are shown and indicate the method is valid for geometric properties of simply or multiply connected plane areas with arbitrary boundaries.

Keywords: Centroid; moment of inertia; product of inertia; arbitrary boundaries; program

Introduction

Geometric properties of an area, including the centroid, the moment of inertia, the polar moment of inertia, and the product of inertia, are important quantities used in mechanics of materials, structural mechanics, fluid mechanics, and machine design, hence their computation by the computer is helpful for the engineer. Wen and Zhang [1] studied geometric properties of an arbitrary triangle and applied to convex polygons. Based on calculating the geometrical characteristic values of a triangle with a vertex on the coordinate origin, Cai [2] suggested a regular method for calculating geometrical and mechanical characteristic values of sections for structural bodies in a shape of prism.

Computational Method for Geometric Properties of an Arbitrary Polygon

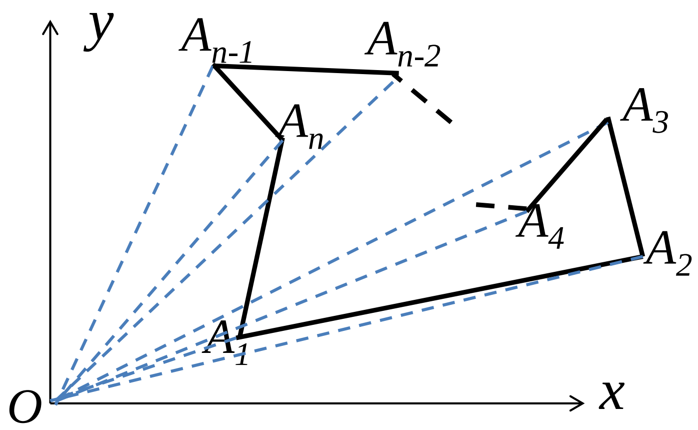

Figure 1. An arbitrary polygon

Consider an arbitrary polygon, shown in Fig. 1, which lies in the *x*-*y* plane. It should be noticed that the vertexes $A_i\,(i=1,2,...,n)$ must be continuously numbered with a counterclockwise direction, and A_{n+1} is assumed to coincide with A_1. x_i and y_i are the coordinates of the vertex A_i. S_i is the algebraic area of ΔOA_iA_{i+1} and can be expressed as Eq. (1). Since the area of a polygon is the sum of S_i, substituting geometric properties of an arbitrary triangle shown in the reference [2] into the general integral definition of geometric properties, we have geometric properties of an arbitrary polygon shown as Eq. (2)- Eq. (5), where x_C and y_C are the coordinates of the centroid C, I_x, I_y, and I_{xy} are the moment of inertia and the product of inertia respectively.

$$S_i = (x_i y_{i+1} - x_{i+1} y_i)/2 \quad (i = 1, 2, \ldots, n) \tag{1}$$

$$x_C = \frac{\sum_{i=1}^{n} S_i \cdot \frac{x_i + x_{i+1}}{3}}{\sum_{i=1}^{n} S_i}, \; y_C = \frac{\sum_{i=1}^{n} S_i \cdot \frac{y_i + y_{i+1}}{3}}{\sum_{i=1}^{n} S_i} \tag{2}$$

$$I_x = \sum_{i=1}^{n} S_i \cdot \frac{y_i^2 + y_i y_{i+1} + y_{i+1}^2}{6} \tag{3}$$

$$I_y = \sum_{i=1}^{n} S_i \cdot \frac{x_i^2 + x_i x_{i+1} + x_{i+1}^2}{6} \tag{4}$$

$$I_{xy} = \sum_{i=1}^{n} S_i \cdot \frac{2(x_i y_i + x_{i+1} y_{i+1}) + x_i y_{i+1} + x_{i+1} y_i}{12} \tag{5}$$

Matlab Program

The Matlab Program for the computation of area, centroid, moment of inertia, polar moment of inertia, and product of inertia for an arbitrary polygon is as follows:

```
clear;clc;
xy=load('coordinates.txt') ;            % vertex coordinate matrix
n=length(xy);
xy=[xy;xy(1,:)];                         % An+1=A1
for i=1:n
s(i)=xy(i,1)*xy(i+1,2)-xy(i+1,1)*xy(i,2);
xci(i)=(xy(i,1)+xy(i+1,1));
yci(i)=(xy(i,2)+xy(i+1,2));
ixi(i)=xy(i,1)^2+xy(i,1)*xy(i+1,1)+xy(i+1,1)^2;
iyi(i)=xy(i,2)^2+xy(i,2)*xy(i+1,2)+xy(i+1,2)^2;
ixyi(i)=2*(xy(i,1)*xy(i,2)+xy(i+1,1)*xy(i+1,2))+xy(i,1)*xy(i+1,2)+xy(i+1,1)*xy(i,2);
end
S=sum(s)/2                          % area of an arbitrary polygon
xc=sum(s.*xci)/area/2/3             % the centroid
yc=sum(s.*yci)/area/2/3             % the centroid
Ix=sum(s.*iyi)/12                   % the moment of inertia about x-axis
Iy=sum(s.*ixi)/12                   % the moment of inertia about y-axis
Ip=Ix+Iy                            % the polar moment of inertia
Ixy=sum(s.*ixyi)/24                 % the product of inertia
```

Numerical Examples

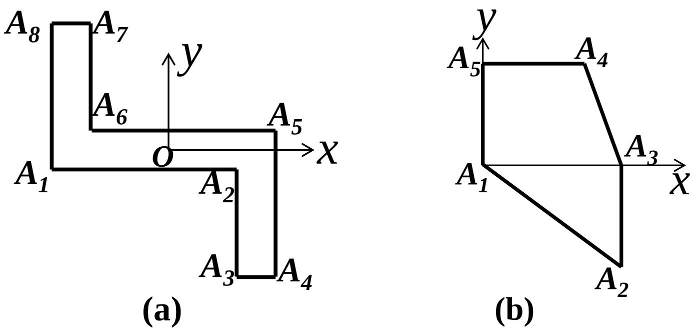

Figure 2. Polygonal cross-sectional areas

In Fig. 2a, the coordinates of the vertexes are $A_1(-300,-50)$, $A_2(200,-50)$, $A_3(200,-350)$, $A_4(300,-350)$, $A_5(300,50)$, $A_6(-200,50)$, $A_7(-200,350)$, and $A_8(-300,350)$, where all the length units are mm. Substituting these coordinates into Eq. (1)- Eq. (5), geometric properties of the polygon can be solved by the above Matlab program. Results are the area $S = 120000\,mm^2$, the centroid $x_C = 0$ and $y_C = 0$, the moment of inertia $I_x = 2.9\text{e}9\,\text{mm}^4$ and $I_y = 5.6\text{e}9\,\text{mm}^4$, the polar moment of inertia $I_O = 8.5\text{e}9\,\text{mm}^4$, and the product of inertia $I_{xy} = \text{-}3\text{e}9\,\text{mm}^4$. In Fig. 2b, the coordinates of the vertexes are $A_1(0,0)$, $A_2(9,-6)$, $A_3(9,0)$, $A_4(6,6)$, and $A_5(0,6)$, where all the length units are mm. Substituting these coordinates into Eq. (1)- Eq. (5), geometric properties of the polygon can be solved by the above Matlab program. Results are the area $S = 72\,mm^2$, the centroid $x_C = 4.625\,mm$ and $y_C = 1\,mm$, the moment of inertia $I_x = 648\,\text{mm}^4$ and $I_y = 1971\,\text{mm}^4$, and the product of inertia $I_{xy} = 81\,\text{mm}^4$. The above results agree with the reference [3].

Computational Method for Geometric Properties of Arbitrary Plane Areas

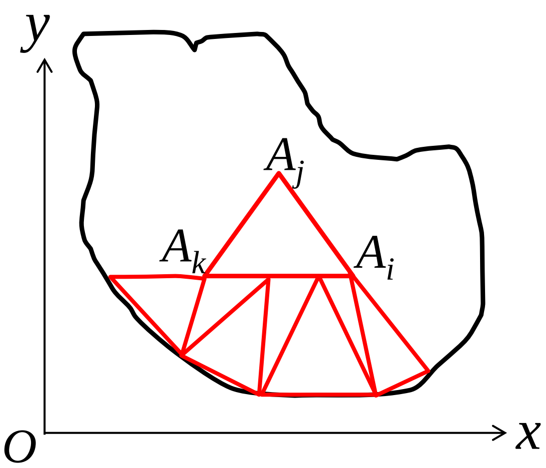

Figure 3. An arbitrary plane area

As shown in Fig. 3, an arbitrary plane area can be easily meshed into a finite number of triangles by the software such as ANSYS, and the coordinates of nodes can be automatically obtained. Substituting geometric properties of an arbitrary triangle shown in the reference [1] into the general integral definition of geometric properties, we have geometric properties of an arbitrary plane area shown as Eq. (6)- Eq. (10), where S_i is the area of meshed triangle

$\Delta A_i A_j A_k$, x_C and y_C are the coordinates of the centroid C, I_x, I_y, and I_{xy} are the moment of inertia and the product of inertia respectively.

$$S_i = \frac{1}{2}\left| x_i(y_j - y_k) + x_j(y_k - y_i) + x_k(y_i - y_j) \right| \tag{6}$$

$$x_C = \frac{\sum_{i=1}^{n} S_i \cdot \frac{x_i + x_j + x_k}{3}}{\sum_{i=1}^{n} S_i}, \; y_C = \frac{\sum_{i=1}^{n} S_i \cdot \frac{y_i + y_j + y_k}{3}}{\sum_{i=1}^{n} S_i} \tag{7}$$

$$I_x = \sum_{i=1}^{n} S_i \cdot \frac{y_i^2 + y_j^2 + y_k^2 + y_i y_j + y_k y_j + y_i y_k}{6} \tag{8}$$

$$I_y = \sum_{i=1}^{n} S_i \cdot \frac{x_i^2 + x_j^2 + x_k^2 + x_i x_j + x_k x_j + x_i x_k}{6} \tag{9}$$

$$I_{xy} = \sum_{i=1}^{n} S_i \cdot \frac{2(x_i y_i + x_j y_j + x_k y_k) + x_i(y_j + y_k) + x_j(y_k + y_i) + x_k(y_i + y_j)}{12} \tag{10}$$

Matlab Program

With the triangular elements and coordinates of nodes obtained by the ANSYS software, the Matlab Program for the computation of area, centroid, moment of inertia, polar moment of inertia, and product of inertia for an arbitrary plane area is as follows:

```
clear;clc;
node=load('node.txt');   % coordinates of nodes
b=load('ele.txt');       % triangular elements
n=length(b);
for m=1:n
  xi=node(b(m,7),2);yi=node(b(m,7),3);
  xj=node(b(m,8),2);yj=node(b(m,8),3);
  xk=node(b(m,9),2);yk=node(b(m,9),3);
s(m)=abs(xi*(yj-yk)+xj*(yk-yi)+xk*(yi-yj));
xci(m)=xi+xj+xk;
yci(m)=yi+yj+yk;
ixi(m)=yi^2+yj^2+yk^2+yi*yj+yj*yk+yk*yi;
iyi(m)=xi^2+xj^2+xk^2+xi*xj+xj*xk+xk*xi;
ixyi(m)=2*(xi*yi+xj*yj+xk*yk)+xi*(yj+yk)+xj*(yk+yi)+xk*(yi+yj);
end
S=sum(s)/2                      % area of an arbitrary plane area
xc=sum(s.*xci)/area/2/3         % the centroid
yc=sum(s.*yci)/area/2/3         % the centroid
Ix=sum(s.*iyi)/12               % the moment of inertia about x-axis
Iy=sum(s.*ixi)/12               % the moment of inertia about y-axis
Ip=Ix+Iy                        % the polar moment of inertia
Ixy=sum(s.*ixyi)/24             % the product of inertia
```

Numerical Examples

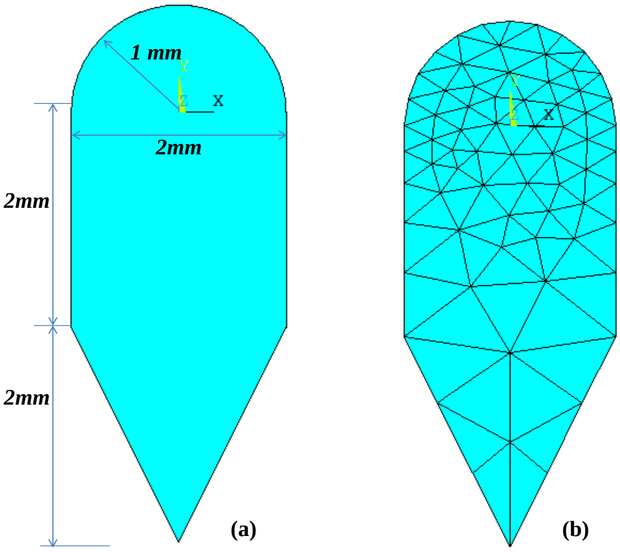

Figure 4. General cross-sectional areas

In Fig. 4, a general cross-sectional area is meshed into 112 triangular elements by the ANSYS software, and the coordinates of nodes can also be obtained by the software. Substituting these coordinates into Eq. (6)- Eq. (10), geometric properties can be solved by the above Matlab program. Results are the area $S = 7.5529\,mm^2$, the centroid $x_C = \text{-}1.1637\text{e-}17\,mm$ and $y_C = \text{-}1.1490\,mm$, the moment of inertia $I_x = 2.0505\,\mathrm{mm}^4$ and $I_y = 20.3838\,\mathrm{mm}^4$, the polar moment of inertia $I_O = 22.4343\,\mathrm{mm}^4$, and the product of inertia $I_{xy} = 1.8619\text{e-}016\,\mathrm{mm}^4$. While both the theoretical values of x_C and I_{xy} should be zero, it indicates numerical computation exits the rounding error.

Conclusions

Eq. (1)- Eq. (5) are exact for geometric properties of arbitrary simply connected polygons with straight edges. Eq. (6)- Eq. (10) can be applied to compute plane areas with arbitrary boundaries. They are easy to be executed by the computer program, however there may exits some little rounding errors. For the multiply connected polygon, it can be considered as a combination of some simply connected polygons and solved by Eq. (1)- Eq. (5), or solved directly by Eq. (6)- Eq. (10).

Acknowledgement

Project Supported by Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant No. KJ1713340).

References

[1] Wen, Z. M., Zhang, L. S. (2001) Using the unit of coordinate to find moment of area, the moment of inertia and the product of inertia, *Journal of Southern Institute of Metallurgy* **22**, 293–295.

[2] Cai, J. B. (1995) The regular method of calculating the geometrical and mechanical characteristic values of prismatic section. *Acta univ. agric. boreali-occidentalis* **23**, 69–73.

[3] Hibbeler, R. C. (2004) *Engineering Mechanics: Statics*, 10th edn, Higher Education Press, Beijing, China.