Nyström extrapolation algorithm for solving delay Volterra integral equations with weakly singular kernel☆

Li Zhang, Jin Huang, Yubin Pan, Hongyan Liu

School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu Sichuan 611731, China

Abstract

This paper concentrates on the numerical solutions of the weakly singular Volterra integral equations with vanishing delay by means of the Nyström method based on the modified rectangular quadrature formula. An extrapolation type arithmetic can be constructed according to the Euler-Maclaurin expansion of error. The algorithm is beneficial to increase the convergence rate of the approach and reaches higher accuracy than the former procedures. Moreover, a posterior error estimate is derived, which can be used to formulate self-adaptive algorithm. Numerical examples demonstrate that this method is effective and applicable.

Keywords: , weakly singular Volterra integral equations, vanishing delay, Nyström method, extrapolation, a posterior error estimate

1. Introduction

In recent years, Volterra integral equations (VIEs) with proportional delay have received a considerable amount of attentions.The VIEs play an important role in the fields of science,engineering and radiative heat transfer problems. There are many numerical techniques to solve these equations. In this paper, we mainly focus on the delay Volterra integral equations with weakly singular kernel as follows:

$$u(t) = g(t) + \int_0^{qt} s^\alpha k(t,s)u(s)ds, t \in [0,T], \tag{1}$$

where g and k are continuous when $t \in [0,T]$, $0 < q < 1, -1 < \alpha < 0$.

In the literature [1], Brunner H.introduced the collocation on graded mesh to solve the weakly singular VIEs. Then,in [2] Chen Y and Tang T assorted the spectral methods to solve weakly singular VIEs with smooth solutions. As to the delay equations, Ali I et al. employed the spectral method for the pantograph-type delay differential equations in [3] and the differential-integral equations with multiple delay in [4]. H.xie.et al.[5] have solved VIEs with vanishing delay by means of the collocation methods. In the recent

☆This work was supported by the National Natural Science Foundation of China (11371079)

Email addresses: `lizhang_137363@163.com` (Li Zhang), `huangjin12345@163.com` (Jin Huang), `yubinpan2014@163.com` (Yubin Pan), `Hongyan_Liu1528@163.com` (Hongyan Liu)

references, emerging a lot of algorithms, as like the new Bernouli wavelet method [6], multistep Legendre pseudo-spectral method [7], the least squares approximation method [8] and [9] Runge-Kutta method.

However, the researchers focus on solving the equations whose kernel are continuous, few of documents pay attention to the situation that the kernels are singular. Some of the methods mentioned above are lower order methods and have the characteristics of needing large amount of calculations and the storage space is large. Then we propose a kind of algorithm for delay VIEs with vanishing delays, which is called the Nyström extrapolation algorithm based on the mid-rectangular formula. By this means,the calculation scheme is simple and high accuracy. Through the form of the assignment the values only occupying small storage space contrasted with the collation methods.

The remainder of this paper as follows:In section 2, we introduce the Nyström methods based on the modify mid-rectangle formula.In section 3, we get some numerical results, which demonstrate this method is efficient and practical. The final section is for concluding remarks.

2. The numerical methods

In this section, we will construct the Nyström methods [10] based on the modify mid-rectangle formula.When the collocation points have choosed, then,the equation (1) is turned into fredholm type. We first consider the integral

$$Q(g) = \int_{\Omega} g(x)dx, \tag{2}$$

where $\Omega \in R^s$, suppose that there exists n-points approximate quadrature formula

$$Q_n(g) = \sum_{j=1}^{n} \omega_j^{(n)} g(x_j^{(n)}), \tag{3}$$

where $x_j^{(n)}$ are the basic quadrature points, $\{\omega_j^{(n)}\}_{j=1}^{n}$ are the corresponding integral weights, which satisfy the condition

$$\sum_{j}^{n} |\omega|_j^{(n)} \leq C, \tag{4}$$

and is a constant and independents of the n . When $n \to \infty, Q_n(g) \to Q(g)$. Consider

$$u(x) - \int_{\Omega} k(x,y)u(y)dy = f(x). \tag{5}$$

By means of the quadrature formula we can deduce the approximate equations

$$u_n(x) - \sum_{j}^{n} \omega_j^{(n)} k(x, x_j^{(n)}) u_n(x_j^{(n)}) = f(x), x \in \Omega, \tag{6}$$

Once $\{u_n(x_i), i = 1, \cdots, n\}$ are solved, by the Nyström interpolation methods, we have

$$u_n(x) = \sum_{j}^{n} \omega_j^{(n)} k(x, x_j^{(n)}) u_n(x_j^{(n)}) + f(x), x \in \Omega, \tag{7}$$

specially,when $\omega_i^{(n)} = h$, it's the mid-rectangle formula. In order to solve the weakly singular kernel, we must make some appropriate changes. For simplicity, we consider the type of linear,as follows:

$$u_i = g(x_i) + h \sum_{j=0}^{i-1} (x_i - x_{j+\frac{1}{2}})^{\alpha} k(x_i, x_{j+\frac{1}{2}}) \frac{(u_j + u_{j+1})}{2}, i = 1, \cdots, N \tag{8}$$

where $(x_i)_{i=1}^{n}$ are collocation points, and the original value $u_0 = g(0)$, Assume that $u \in C^2[0,T]$ is the solution of (1) , and the kernel satisfies Lipschitz continuity condition with constant s and t, Then, the expansion of errors is

$$u_i - u(x_i) = T_0^M(x_i) h^{2+\alpha} + O(h^2), i = 1, \cdots, N. \tag{9}$$

Therefore, in order to achieve high accuracy, we can utilize the technology of extrapolation with the aid of the error of gradual expansion. Furthermore, we consider a posteriori error estimate.First, we execute Richardson extrapolation with $h^{2+\alpha}$, get

$$u_i^h = \frac{2^{2+\alpha} u_i^{(\frac{h}{2})} - u_i^h}{2^{2+\alpha} - 1} = u(x_i) + O(h^2), i = 1, \cdots, N. \tag{10}$$

Now, we will construct the concrete scheme on delay integral equations with weakly singular kernel.First, we divide $I \in [0,T]$ into several intervals such that the stepsizes $h = \frac{T}{N}$,$t_i = ih, i = 0, 1, \cdots, N$. Let $t = t_i$, we get

$$\begin{aligned} u(t_i) &= g(t_i) + \int_0^{qt_i} k(t_i, s) u(s) ds \\ &= g(t_i) + \int_0^{t_{[qi]}} k(t_i, s) u(s) ds + \int_{t_{[qi]}}^{qt_i} k(t_i, s) u(s) ds \\ &= g(t_i) + I_1 + I_2, \end{aligned} \tag{11}$$

where $[qi]$ denote the integer part of qi. In order to structure the rectangular quadrature algorithm, only using modify mid-rectangle formula on I_1 and I_2,

$$I_1 = \int_0^{t_{[qi]}} k(t_i, s) u(s) ds \approx h \sum_{k=0}^{[qi]-1} k(t_i, t_{k+\frac{1}{2}}) u(t_{k+\frac{1}{2}}), \tag{12}$$

and

$$I_2 = \int_{t_{[qi]}}^{qt_i} k(t_i, s) u(s) ds \approx (qt_i - t_{[qi]}) k(t_i, \frac{qt_i + t_{[qi]}}{2}) u(\frac{qt_i + t_{[qi]}}{2}). \tag{13}$$

Taking into account $u(qt_i)$ are not the node values, so we can turn to the linear interpolation approximation with values of the adjacent points $u(t_{[qi]})$ and $u(t_{[qi]}+1)$, with $t_{[qi]} \leq qt_i \leq t_{[qi]+1}$. Then, there exists $\beta_i \in [0,1]$ such that $qt_i = \beta_i t_{[qi]} + (1-\beta_i)t_{[qi]+1}$ are established, where $\beta_i = 1 + [qi] - qi$. Clearly, we can get

$$u(\frac{qt_i + t_{[qi]}}{2}) = \frac{1+\beta}{2}u(t_{[qi]}) + \frac{1-\beta}{2}u(t_{[qi]+1}), \tag{14}$$

and $u(t_{k+\frac{1}{2}}) = \frac{1}{2}u(t_k) + \frac{1}{2}u(t_{k+1})$.

3. The numerical results

Two examples have been presented to show the efficiency of the Nyström extrapolation algorithms.For simplicity,we design a set of grids on the interval I ,we take $N = 40, 80$ and $h = \frac{1}{N}$ in the following two examples.

Example 1. Let us consider the second kind linear volterra integral equation of

$$u(t) = t^{\frac{5}{2}} - 0.243t^5 - 0.164025t^4 + \int_0^{qt} s^{-\frac{1}{2}}(t^2+s)u(s)ds, t \in [0,T], \tag{15}$$

with $q = 0.9, T = 1$,the initial value $u(0) = g(0) = 0$ and the analytical solution is $u(t) = t^{\frac{5}{2}}$. we denote the approximate values by u_h, The absolute errors $e_h(t) = \ u(t) - u_h(t)|, t \in [0,1]$ and the posteriori error have been presented in the Table.1.Futhermore, the $h^{2+\alpha}$ extrapolation values and the h^2 extrapolation values have also been shown in the Table.1. indicate that our algorithm provides high accuracy results for the weakly singular at endpoint.

Table 1: The results of example 1

t	e_h	$e_{\frac{h}{2}}$	$h^{2+\alpha}$ extrapolation	h^2 extrapolation	posteriori error
0.2	3.96e-06	1.01e-06	6.05e-07	2.79e-07	1.61e-06
0.4	2.42e-05	6.11e-06	3.82e-06	1.72e-07	9.93e-06
0.6	7.69e-05	1.91e-05	1.25e-05	2.12e-07	3.16e-05
0.8	1.87e-04	4.65e-05	3.01e-05	8.97e-07	7.67e-05
1	3.95e-04	9.87e-05	6.33e-05	1.26e-07	1.62e-04

Example 2. This example is to further demonstrate the effectiveness of the Nyström method based on the modify mid-rectangle formula.We consider the equation as follows:

$$u(t) = t - \frac{2}{3} \times 0.8^{\frac{3}{2}} t^{\frac{3}{2}} + \int_0^{qt} s^{-\frac{1}{2}}u(s)ds, t \in [0,T], \tag{16}$$

the analytical solution is $u(t) = t$,we take $q = 0.8, T = 1$ and the initial value $u(0) = g(0) = 0$ .We use the Nyström extrapolation scheme for solving (16).From the Table 2, we see that the approximate solution obtained by the $h^{2+\alpha}$ extrapolation and the h^2 extrapolation are effective on the whole interval.

Table 2: The analysis of example 2

t	e_h	$e_{\frac{h}{2}}$	$h^{2+\alpha}$ extrapolation	h^2 extrapolation	posteriori error
0.2	3.2907e-04	1.3047e-04	2.1853e-05	5.0705e-06	1.0862e-04
0.4	4.5591e-04	1.7906e-04	2.7650e-05	6.3606e-06	1.5141e-04
0.6	5.7579e-05	2.2550e-04	3.3926e-05	7.9592e-06	1.9158e-04
0.8	6.9634e-04	2.7240e-04	4.0548e-05	9.8280e-06	2.3186e-04
1	8.2037e-04	3.2066e-04	4.7359e-05	1.1556e-05	2.7330e-04

4. Conclusion

From the above tables,we can conclude that the Nyström method based on the modify mid-rectangle formula is an efficient and accurate numerical technique for Volterra integral equations of second with weakly singular kernel. In particular, this approach is novel for various problem with cauchy singular kernel and the hyper-singular delay integral equations.

5. references

[1] Brunner H. The numerical solutions of weakly singular Volterra integral equations by collocation on graded mesh. Math Comp, 1985, 45: 417-437

[2] Chen Y, Tang T. Spectral methods for weakly singular Volterra integral equations with smooth solutions. J Comput Appl Math, 2009, 233: 938C950

[3] Ali I, Brunner H, Tang T. A spectral method for pantograph-type delay differential equations and its convergence analysis. J Comput Math, 2009, 27: 254C265

[4] Ali I, Brunner H, Tang T. Spectral method for pantograph-type differential and integral equations with multiple delays. Front Math China, 2009, 4: 49C61

[5] H. Xie , R. Zhang , H. Brunner , Collocation methods for general Volterra functional integral equations with vanishing delays, SIAM J. Sci. Comput. 33 (2011)3303C3332.

[6] P.K.Sahu,S.Saha Ray,A new Bernoulli wavelet method for accurate solutions of nonlinear fuzzy Hammerstein-Volterra delay integral equations,FSS,399(2017)131-144.

[7] Zhang Xiao-yong,Convergence analysis of the multistep Legendre pseudo-spectral method for Volterra integral equations with vanishing delays,J.Comput.Appl.Math,321(2017)284-301.

[8] Maryam Mosleh,Mehmood Otadi,Least squares approximation method for the solution of Hammerstein-Volterra delay integral equations,AMC,258(2015)105-110.

[9] Saberi-Nadiafi,M.Tamamagar,A generalized block-by-block method for solving linear Volterra integral equations,Appl.Math.Comput.188(2017)1969-1974.

[10] J.Huang,T.Lü,High accuracy algorithm of integral eqautions[M],Science press,2013.