A complex variable interpolating meshless method for two-dimensional transient heat conduction problem

***Yajie Deng[1], †Xiaoqiao He[1,2]**

[1] Department of Architecture and Civil Engineering, City University of Hong Kong, Hong Kong
[2] Center for Advanced Structural Materials, City University of Hong Kong Shenzhen Research Institute, Shenzhen 518057, China

*Presenting author: yjdeng7-c@my.cityu.edu.hk
†Corresponding author: bcxqhe@cityu.edu.hk

Abstract

In this paper, on the basis of complex variable moving least squares approximations, a new complex variable moving least squares interpolating (CVMLSI) method is proposed. In this method, a complete basis function and singular weight function are introduced to form the new basis function through orthogonalization process. Then a new interpolating shape function is derived, which satisfy the property of Kronecker δ function.

Combining the CVMLSI method with the weak integral form of the two-dimensional transient heat conduction problem, a complex variable element free Galerkin interpolating (CVEFGI) method for transient heat conduction problem is obtained. Due to the fact that the essential boundary conditions can be applied directly, the final discrete matrix equation is more concise than that in the non-interpolating complex variable element free Galerkin method. Finally, a numerical example is presented to illustrate the advantages of the CVEFGI method.

Keywords: Complete basis function, interpolating meshless method, Kirchhoff plates

Introduction

As a numerical tool, the meshless method [1] has developed widely in engineering analysis. Different with the traditional mesh based numerical methods, such as the finite element method, and boundary element method [2][3], the meshless method is built on a series of discrete nodes. So when using the meshless method to solve some special and complicated problems, such as nonlinear large deformation of polymer gel, crack propagation problem and so on, the re-meshing techniques are not necessary during the computing process [4][5].

One of the most common used methods to build the trial function is the moving least squares (MLS) approximation [6]. On the base of the MLS approximation, Belytschko presented the element free Galerkin (EFG) method [7]. With the development of the meshless method, a variety of complex variable moving least squares method approximations were proposed on the foundation of the MLS approximation. In these complex variable moving least squares approximations, the basis functions $\boldsymbol{p}^{\mathrm{T}}(z)=(1,z)$ and $\overline{\boldsymbol{p}}^{\mathrm{T}}(z)=(1,\bar{z})$ are used to construct the trial functions [8][9]. Then the trial function of a two-dimensional problem can be formed with a one-dimensional basis function, which leads to the complex variable moving least squares approximations have higher efficiency.

However, the basis functions above mentioned are not complete basis functions, which can not express all functions in the problem domain and may reduce the computing accuracy. Besides, in most of the complex variable moving least squares approximations, the obtained shape functions can not satisfy the property of Kronecker δ function. In the meshless method built on these approximations, special techniques are useful to apply the essential boundary conditions, such as Lagrange multiplier and penalty methods [1][10].

Trying to solve above two problems, in this paper, we introduce a complete basis function $\boldsymbol{p}^{\mathrm{T}}(z)=(1,z,\bar{z})$ and the singular weight function. Then reference the idea presented by Ren [11], improve the basis function with orthogonalization process to get the new interpolating shape function. Then a new complex variable moving least squares interpolating (CVMLSI) method is presented. Combining the CVMLSI method with the weak integral form of the two-dimensional transient heat conduction problem, a complex variable element free Galerkin interpolating (CVEFGI) method for heat conduction problem is obtained and the final matrix equation is derived. Finally, a numerical example is solved to validate the advantages of the CVEFGI method compared with non-interpolating complex variable element free Galerkin method.

Methodology

In this part, the CVMLSI method is introduced. According to the improved complex variable moving least squares (ICVMLS) approximation presented by Bai [9], the trial function can be expressed as

$$u^h(z)=u_1^h(z)+\mathrm{i}u_2^h(z)=\sum_{i=1}^{m}p_i(z)a_i(z)=\boldsymbol{p}^{\mathrm{T}}(z)\boldsymbol{a}(z),\qquad (z=x_1+\mathrm{i}x_2\in\Omega), \tag{1}$$

where the $\boldsymbol{p}^T(z)=(p_1(z),p_2(z),\ldots,p_m(z))$ is the complete basis function vector. In the two-dimensional domain, the linear and quadratic basis function vectors are shown as

$$\boldsymbol{p}^{\mathrm{T}}(z)=(1,z,\bar{z}), \tag{2}$$

$$\boldsymbol{p}^{\mathrm{T}}(z)=(1,z,\bar{z},z^2,\bar{z}^2,z\bar{z}). \tag{3}$$

The local approximation at point z can be expressed as

$$u^h(z,\hat{z})=\sum_{i=1}^{m}p_i(\hat{z})a_i(z)=\boldsymbol{p}^{\mathrm{T}}(\hat{z})\boldsymbol{a}(z), \tag{4}$$

where $\hat{z}$ is the node whose influence domain covers the point z.

Then, combining the singular weight function to make the following improvement on the space $span(p_1,p_2,...,p_m)$, $p_1(z)\equiv 1$ is normalized as [12]

$$\beta_z^{(1)}(z)=\frac{p_1}{\|p_1\|_z}=\frac{1}{[\sum_{I=1}^{n}w(z-z_I)]^{\frac{1}{2}}}, \tag{5}$$

let $p_i(z)$ $(i=2,3,...,m)$ be orthogonal to $\beta_z^{(1)}(z)$, then we can gain that

$$\begin{aligned} b_z^{(i)}(z)&=p_i(z)-(p_i(z),\beta_z^{(1)}(z))_z\beta_z^{(1)}(z)\\ &=p_i(z)-\frac{\sum_{I=1}^{n}p_i(z_I)w(z-z_I)}{\sum_{I=1}^{n}w(z-z_I)} \end{aligned}$$

$$= p_i(z) - \sum_{I=1}^{n} p_i(z_I) v(z - z_I)\,,\ (i = 2,3,...,m)\,, \tag{6}$$

that is

$$b_z^{(i)}(z) = p_i(z) - p_i^{(s)}(z) = p_i(z) - \boldsymbol{v}^{\mathrm{T}}(z)\boldsymbol{p}_i\,. \tag{7}$$

Then, the $\beta_z^{(1)}(z), b_z^{(2)}(z),...,b_z^{(m)}(z)$ are the new basis function, and the new interpolating function is

$$u^h(z) = (u, \beta_z^{(1)}(z))_z \beta_z^{(1)}(z) + \sum_{i=2}^{m} a_i(z) b_z^{(i)}(z)\,, \tag{8}$$

i.e.

$$u^h(z) = \boldsymbol{v}^{\mathrm{T}}(z)\boldsymbol{u} + \boldsymbol{b}^{\mathrm{T}}(z)\boldsymbol{a}(z)\,, \tag{9}$$

where

$$\boldsymbol{v}(z) = (v(z - z_1), v(z - z_2),..., v(z - z_n))^{\mathrm{T}}\,, \tag{10}$$

$$\boldsymbol{u} = (u(z_1), u(z_2),..., u(z_n))^{\mathrm{T}}\,, \tag{11}$$

$$\boldsymbol{a}(z) = (a_2(z), a_3(z),..., a_m(z))^{\mathrm{T}}\,, \tag{12}$$

$$\boldsymbol{b}(z) = (b_z^{(2)}(z), b_z^{(3)}(z),..., b_z^{(m)}(z))^{\mathrm{T}}\,. \tag{13}$$

In Eq. (10), $v(z - z_i)$ is the normalized weight function when consider the special condition $m = 1$, which is a weighted average of the function values at node z_I in the influence domain of point z. The expression form is

$$v(z - z_i) = \frac{w(z - z_i)}{\sum_{I=1}^{n} w(z - z_I)}\,, \quad (i = 1,2,...,n)\,. \tag{14}$$

Using the functional J in the ICVMLS approximation [9] and ensure the J is minimum, the corresponding unknown coefficient vector is obtained as,

$$\boldsymbol{a}(z) = \boldsymbol{A}_z^{-1}(z)\boldsymbol{B}_z(z)\boldsymbol{u}\,, \tag{15}$$

where

$$\boldsymbol{A}_z(z) = \overline{\boldsymbol{C}}_z^{\mathrm{T}} \boldsymbol{W}(z) \boldsymbol{C}_z\,, \tag{16}$$

$$\boldsymbol{B}_z(z) = \overline{\boldsymbol{C}}_z^{\mathrm{T}} \boldsymbol{W}(z)\,, \tag{17}$$

$$\boldsymbol{C}_z = \begin{bmatrix} b_z^{(2)}(z_1) & b_z^{(3)}(z_1) & \cdots & b_z^{(m)}(z_1) \\ b_z^{(2)}(z_2) & b_z^{(3)}(z_2) & \cdots & b_z^{(m)}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_z^{(2)}(z_n) & b_z^{(3)}(z_n) & \cdots & b_z^{(m)}(z_n) \end{bmatrix}, \tag{18}$$

$$\boldsymbol{W}(z) = \begin{bmatrix} w(z - z_1) & 0 & \cdots & 0 \\ 0 & w(z - z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(z - z_n) \end{bmatrix}. \tag{19}$$

Substituting Eq. (15) into Eq. (9), we can obtain

$$u^h(z) = \boldsymbol{v}^{\mathrm{T}}(z)\boldsymbol{u} + \boldsymbol{b}^{\mathrm{T}}(z)\boldsymbol{A}_z^{-1}(z)\boldsymbol{B}_z(z)\boldsymbol{u} = \boldsymbol{\Phi}(z)\boldsymbol{u} = \sum_{I=1}^{n} \Phi_I(z) u(z_I)\,, \tag{20}$$

where $\boldsymbol{\Phi}(z)$ is the new interpolating shape function

$$\boldsymbol{\Phi}(z) = (\Phi_1(z), \Phi_2(z), \ldots, \Phi_n(z)) = \boldsymbol{v}^{\mathrm{T}}(z) + \boldsymbol{b}^{\mathrm{T}}(z)\boldsymbol{A}_z^{-1}(z)\boldsymbol{B}_z(z) , \tag{21}$$

and

$$u_1^h(z) = \mathrm{Re}[\boldsymbol{\Phi}(z)\boldsymbol{u}] = \mathrm{Re}[\sum_{I=1}^{n} \Phi_I(z) u(z_I)] , \tag{22}$$

$$u_2^h(z) = \mathrm{Im}[\boldsymbol{\Phi}(z)\boldsymbol{u}] = \mathrm{Im}[\sum_{I=1}^{n} \Phi_I(z) u(z_I)] . \tag{23}$$

The following singular weight function is used as [13]

$$w(z - z_I) = \begin{cases} \dfrac{\rho^2}{|z - z_I|^2}\left(1 - \dfrac{\rho}{|z - z_I|}\right)^2 & |z - z_I| \le \rho \\ 0 & |z - z_I| \ge \rho \end{cases} , \tag{24}$$

where ρ is the radius of the influence domain of the point z. This singular weight function can satisfy general properties of other weight functions, such as the cubic and quartic spline weight function [1].

This is the CVMLSI method. Because the new complete basis function and the singular weight function is used to construct the trial function, the CVMLSI method has higher accuracy and the shape function obtained from this method can satisfy the property of Kronecker δ function.

Numerical Example

The CVMLSI method is used to solve the two-dimensional transient heat conduction problem. The governing equation is [14]

$$\rho c \frac{\partial T(\boldsymbol{x},t)}{\partial t} - \frac{\partial}{\partial x_1}\left(k_1 \frac{\partial T(\boldsymbol{x},t)}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(k_2 \frac{\partial T(\boldsymbol{x},t)}{\partial x_2}\right) - Q = 0 ,$$

$$(\boldsymbol{x} = (x_1, x_2) \in \Omega), \tag{25}$$

with the following boundary conditions and initial condition

$$T(\boldsymbol{x},t) - \overline{T} = 0 , \qquad (\boldsymbol{x} \in \Gamma_1), \tag{26}$$

$$k_1 \frac{\partial T(\boldsymbol{x},t)}{\partial x_1} n_1 + k_2 \frac{\partial T(\boldsymbol{x},t)}{\partial x_2} n_2 - \overline{q} = 0 , \qquad (\boldsymbol{x} \in \Gamma_2), \tag{27}$$

$$k_1 \frac{\partial T(\boldsymbol{x},t)}{\partial x_1} n_1 + k_2 \frac{\partial T(\boldsymbol{x},t)}{\partial x_2} n_2 - h(T_a - T(\boldsymbol{x},t)) = 0 , \quad (\boldsymbol{x} \in \Gamma_3), \tag{28}$$

$$T(\boldsymbol{x},0) = T_0 , \tag{29}$$

where $T(\boldsymbol{x},t)$ is the temperature field function, Q is the internal heat generation per unit volume, $\overline{T}$ is the given temperature on the boundary Γ_1, $\overline{q}$ is the given density of the teat flux on the boundary Γ_2 and T_0 is the given initial temperature.

When build the CVEFGI method for the two-dimensional transient heat conduction problem, using the CVMLSI method to disperse the spatial domain and using the difference method to disperse the time, then we can obtain the final matrix equation

$$(\frac{\boldsymbol{C}}{\Delta t}+\theta\overline{\boldsymbol{K}})\boldsymbol{T}_{t+\Delta t}=(\frac{\boldsymbol{C}}{\Delta t}-(1-\theta)\overline{\boldsymbol{K}})\boldsymbol{T}_t+\theta\overline{\boldsymbol{F}}_{t+\Delta t}+(1-\theta)\overline{\boldsymbol{F}}_t\,, \tag{30}$$

where Δt is the time step,

$$\boldsymbol{C}=\int_\Omega \tilde{\boldsymbol{\Phi}}^{\mathrm{T}}(z)\rho c\tilde{\boldsymbol{\Phi}}(z)\mathrm{d}\Omega\,, \tag{31}$$

$$\overline{\boldsymbol{K}}=\boldsymbol{K}+\boldsymbol{H}\,, \tag{32}$$

$$\overline{\boldsymbol{F}}=\boldsymbol{F}^{(1)}+\boldsymbol{F}^{(2)}+\boldsymbol{F}^{(3)}\,, \tag{33}$$

$$\boldsymbol{K}=\int_\Omega \boldsymbol{B}^{\mathrm{T}}(z)\tilde{\boldsymbol{k}}\boldsymbol{B}(z)\mathrm{d}\Omega\,, \tag{34}$$

$$\boldsymbol{H}=\int_{\Gamma_3} \tilde{\boldsymbol{\Phi}}^{\mathrm{T}}(z)h\tilde{\boldsymbol{\Phi}}(z)\mathrm{d}\Gamma\,, \tag{35}$$

$$\boldsymbol{F}^{(1)}=\int_\Omega \tilde{\boldsymbol{\Phi}}^{\mathrm{T}}(z)Q\mathrm{d}\Omega\,, \tag{36}$$

$$\boldsymbol{F}^{(2)}=\int_{\Gamma_2} \tilde{\boldsymbol{\Phi}}^{\mathrm{T}}\overline{q}\mathrm{d}\Gamma\,, \tag{37}$$

$$\boldsymbol{F}^{(3)}=\int_{\Gamma_3} \tilde{\boldsymbol{\Phi}}^{\mathrm{T}}(z)hT_a\mathrm{d}\Gamma\,. \tag{38}$$

This is the CVEFGI method for the two-dimensional transient heat conduction problem. Compared with the improved complex variable element-free Galerkin (ICVEFG) method based on the ICVMLS approximation, the above expressions of matrices are more concise [14].

For a specific two-dimensional transient heat conduction problem with a square domain, the control equation is

$$\frac{\partial T(x_1,x_2,t)}{\partial t}-\frac{\partial^2 T(x_1,x_2,t)}{\partial x_1^2}-\frac{\partial^2 T(x_1,x_2,t)}{\partial x_2^2}+2=0\,,$$

$$x_1\in[0,\pi],\quad x_2\in[0,\pi], \tag{39}$$

and the essential boundary conditions are

$$T(0,x_2,t)=0\,, \tag{40}$$

$$T(\pi,x_2,t)=\pi^2\,, \tag{41}$$

$$T(x_1,0,t)=x_1^2\,, \tag{42}$$

$$T(x_1,\pi,t)=x_1^2\,. \tag{43}$$

The initial condition is

$$T(x_1,x_2,0)=x_1^2+\sin(x_1)\sin(x_2)\,, \tag{44}$$

and the analytical solution is

$$T(x_1,x_2,t)=x_1^2+e^{-2t}\sin(x_1)\sin(x_2)\,. \tag{45}$$

In the CVEFGI method, the linear basis function and 4×4 Gauss points are used. Try to make error analysis, the relative error is

$$e=\left|\frac{T_i^{num}-T_i^{exact}}{T_i^{exact}}\right|, \tag{46}$$

where T_i^{num} is the numerical solution and T_i^{exact} is the analytical solution.

In this example, 15×15 nodes are distributed uniformly in the square domain. The scaling parameter is $d_{\max}=2$, and the time step is $\Delta t=0.001s$. For the ICVEFG method, the penalty factor is $\alpha=1.0\times10^{4}$.

Fig. 1 and Fig. 2 compare the solutions of heat distributions at $x_2=\pi/2$ and $x_1=\pi/2$ respectively with different times. The numerical solutions obtained from the CVEFGI method and the ICVEFG method are in good agreement with the analytical solutions at different times. Fig. 3 and Fig. 4 show the relative errors of the CVEFGI method and the ICVEFG method at different times at $x_2=\pi/2$ and $x_1=\pi/2$ respectively. We can see compared with the ICVEFG method, the CVEFGI method has higher computing accuracy, especially on the borders.

Besides due to the essential boundary conditions can be applied directly in the CVEFGI method, the final discrete equation of this transient heat conduction problem is simpler than that in the ICVEFG method. And there is need to choose suitable penalty factor or Lagrange multiplier which will save more computing time than the ICVEFG method.

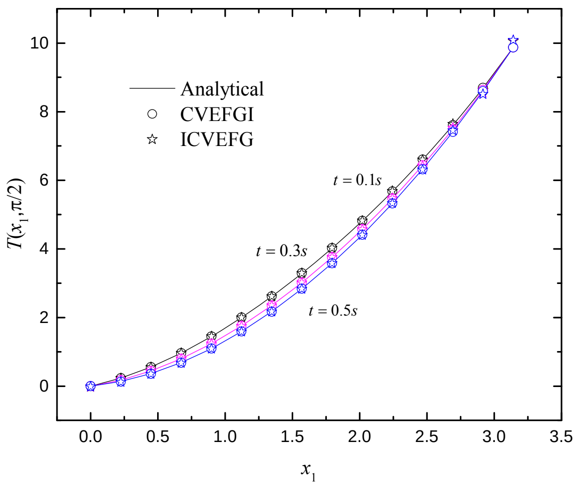

Figure 1. Heat distributions at $x_2=\pi/2$ with different times

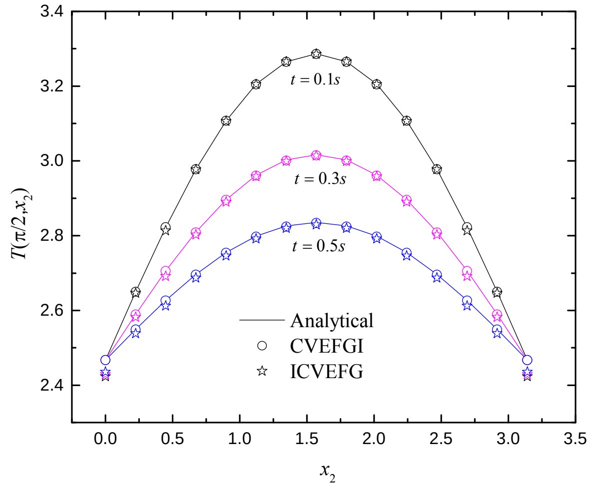

Figure 2. Heat distributions at $x_1 = \pi/2$ with different times

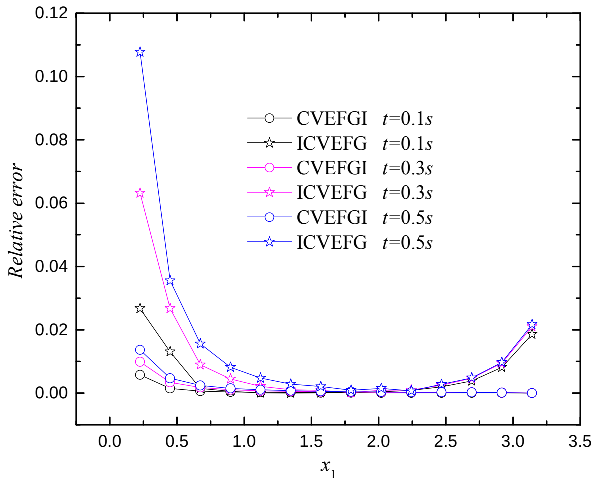

Figure 3. Relative errors at $x_2 = \pi/2$ with different times

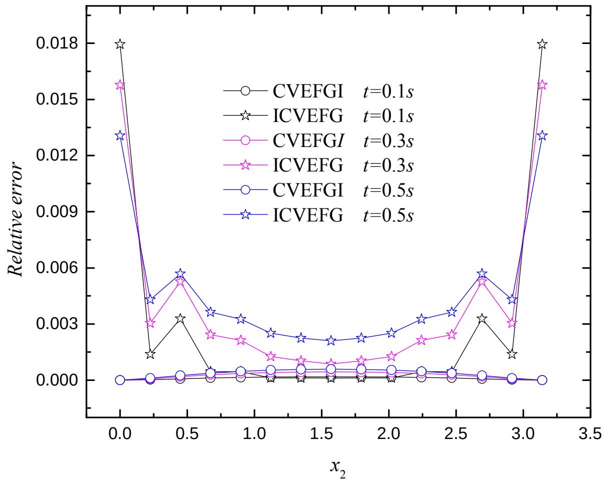

Figure 4. Relative errors at $x_1 = \pi/2$ with different times

Conclusions

In this paper, a complete basis function and singular weight function are introduced to derive the new shape function, and then the CVMLSI method is presented. In the CVMLSI method, the new shape function can satisfy the Kronecker δ function. So compared with the ICVMLS approximation with non-interpolating shape function, the CVMLSI method has higher accuracy. Combining the weak integral form of the two-dimensional transient heat conduction problem and the CVMLSI method, the CVEFGI method for the heat conduction problem is obtained. In the CVEFGI method, because the essential boundary conditions can be applied directly, the final matrix equation is more concise and it is unnecessary to choose appropriate penalty factor. The numerical example shows that the CVEFGI method is more accurate and efficient than the ICVEFG method.

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