

Analytical Computation for Turbulence and Enstrophy in the Taylor Green

Vortex Model

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Abstract

Using a classic example proposed by G. I. Taylor, we reconsider through the use of computer algebra, the mathematical analysis of a fundamental process in turbulent flow, namely: How do large scale eddies evolve into smaller scale ones to the point where they are effectively absorbed by viscosity? The explicit symbolic series solution of this problem, even for cleverly chosen special cases, requires daunting algebra, and so numerical methods have become quite popular. Yet an algebraic approach can provide substantial insight, especially if it can be pursued with modest human effort. The specific example we use dates to 1937 when Taylor and Green first published a method for explicitly computing successive approximations to formulas describing the three-dimensional evolution over time of what is now called a Taylor-Green vortex. With the aid of a computer algebra system, we have duplicated Taylor and Green's efforts and obtained more detailed time-series results. We have extended their approximation of the energy dissipation from order 5 in time to order 8.

Keywords: nonlinear equations, symbolic calculations, vortices

Introduction

The fundamental dynamical mechanism involved in homogeneous three dimensional turbulent flows is the enhancement of vorticity by vortex-line stretching and the consequent production of small-scale eddies. This process controls the turbulent-energy dynamics and hence the global structure and evolution of the flow. A model of this process is given by the Taylor-Green vortex ref [1], which is perhaps the simplest system in which to study the generation of small scales and the resulting turbulence. In the past this flow has been used to study such questions as: (i) enhancement of vorticity by vortex line stretching (ii) approach to isotropy of the small scales ref [2] (iii) possible singular behaviour of solutions of the Euler equations ref [3] and there are other issues of this flow discussed in ref [4-7].

Series Derivation and Computer Extension

The flow discussed at length in this paper is that for which the initial flow has two-dimensional streamlines, but the flow is three-dimensional for all $t > 0$.

This section is an initial statement and development of Taylor's model [1]. The initial flow of an incompressible fluid is chosen as represented by the following equations:

$$u = A \cos(ax) \sin(by) \sin(cz)$$

$$v = B \sin(ax) \cos(by) \sin(cz)$$

$$w = C \sin(ax) \sin(by) \cos(cz)$$

The Taylor-Green vortex is that three-dimensional flow that develops from the single-Fourier mode initial condition. Using the fact that ρ , the density, is constant for an incompressible fluid, and the equation of continuity:

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0 \quad (1)$$

We can derive the following consistency equations for our model:

$$u_x + v_y + w_z = 0$$

$$Aa + Bb + Cc = 0$$

The equations of motion are:

$$\begin{aligned} -u_t &= uu_x + vv_y + ww_z + \frac{P_x}{\rho} \\ -v_t &= uv_x + vv_y + ww_z + \frac{P_y}{\rho} \\ -w_t &= uw_x + vw_y + ww_z + \frac{P_z}{\rho} \end{aligned} \quad (2)$$

$$-\frac{1}{\rho} \nabla^2 P = u_x^2 + v_y^2 + w_z^2 + 2(v_x u_y + w_y v_z + u_z w_x) \quad (3)$$

It is possible to solve these equations (1), (2) and (3) as a regular expansion in the time

$$\begin{aligned} u &= \sum_{i=0}^{\infty} U_i t^i \\ v &= \sum_{i=0}^{\infty} V_i t^i \\ w &= \sum_{i=0}^{\infty} W_i t^i \\ \left(\frac{P}{\rho}\right) &= \sum_{i=0}^{\infty} \left(\frac{P}{\rho}\right)_i t^i \end{aligned} \quad (4)$$

We assume $A=a=b=c=1$ and $B=-1$, which leads to:

$$\begin{aligned} U_0 &= \cos(x) \sin(y) \sin(z) \\ V_0 &= -\sin(x) \cos(y) \sin(z) \\ W_0 &= 0 \end{aligned}$$

Then substituting (4) in the equations (1), (2) and (3) and equating like powers of time t yields this sequence of successive linear equations, together with boundary conditions we obtain:

$$\begin{aligned} U_1 &= -\frac{1}{8} \sin(2x) \cos(2z) & V_1 &= -\frac{1}{8} \sin(2y) \cos(2z) \\ W_1 &= \frac{1}{8} \sin(2z) \cos(2x) + \frac{1}{8} \sin(2z) \cos(2y) \end{aligned}$$

$$\begin{aligned}
\left(\frac{P}{\rho}\right)_1 &= \frac{1}{44} \sin(x) \sin(3y) \sin(z) - \frac{1}{44} \sin(3x) \sin(y) \sin(z) \\
U_2 &= -\frac{15}{352} \sin(3y) \sin(z) \cos(x) - \frac{5}{176} \sin(y) \sin(z) \cos(3x) + \frac{1}{32} \sin(y) \sin(3z) \cos(x) \\
&\quad - \frac{1}{32} \cos(x) \sin(y) \sin(z) + \frac{1}{32} \sin(y) \sin(3z) \cos(3x) \\
V_2 &= \frac{15}{352} \sin(z) \sin(3x) \cos(y) + \frac{5}{176} \sin(z) \sin(x) \cos(3y) - \frac{1}{32} \sin(3z) \sin(x) \cos(y) \\
&\quad + \frac{1}{32} \sin(x) \cos(y) \sin(z) - \frac{1}{32} \sin(3z) \sin(x) \cos(3y) \\
W_2 &= -\frac{15}{352} \sin(x) \sin(3y) \cos(z) + \frac{15}{352} \sin(3x) \sin(y) \cos(z) - \frac{1}{32} \sin(3x) \sin(y) \cos(3z) \\
&\quad + \frac{1}{32} \sin(x) \sin(3y) \cos(3z) \\
\left(\frac{P}{\rho}\right)_2 &= \frac{1}{128} \cos(2x) \cos(2y) \cos(4z) + \frac{21}{1408} \cos(2x) \cos(2y) + \frac{21}{1408} \cos(2y) + \frac{21}{1408} \cos(2x) + \\
&\quad \frac{1}{8448} \cos(2x) \cos(4y) \cos(2z) - \frac{1}{1408} \cos(2x) \cos(4y) - \frac{1}{220} \cos(4y) \cos(2z) \\
&\quad - \frac{5}{264} \cos(2x) \cos(2y) \cos(2z) + \frac{1}{8448} \cos(4x) \cos(2y) \cos(2z) - \frac{1}{220} \cos(2z) \cos(4x) \\
&\quad - \frac{1}{1408} \cos(4x) \cos(2y) - \frac{17}{2816} \cos(2y) \cos(2z) - \frac{17}{2816} \cos(2x) \cos(2z) + \frac{1}{128} \cos(4z) \\
&\quad + \frac{31}{2816} \cos(4x) + \frac{31}{2816} \cos(4y) \\
U_3 &= \frac{1}{384} \cos(2y) \sin(4x) \cos(4z) - \frac{19}{3168} \cos(2y) \cos(2z) \sin(2x) - \frac{1}{198} \cos(2y) \cos(2z) \sin(4x) \\
&\quad + \frac{49}{12672} \sin(2x) \cos(2z) \cos(4y) - \frac{1}{192} \sin(4x) \cos(4z) - \frac{1}{192} \sin(2x) \cos(4z) - \frac{1}{192} \sin(2x) \cos(4y) \\
&\quad + \frac{27}{3520} \cos(2z) \sin(4x) + \frac{1}{384} \cos(2y) \sin(4x) + \frac{43}{4224} \sin(2x) \cos(2z) + \frac{1}{96} \cos(2y) \sin(2x) \cos(4z) \\
V_3 &= \frac{1}{384} \cos(2x) \sin(4y) \cos(4z) - \frac{19}{3168} \cos(2x) \cos(2z) \sin(2y) - \frac{1}{198} \cos(2x) \cos(2z) \sin(4y) \\
&\quad + \frac{49}{12672} \sin(2y) \cos(2z) \cos(4x) - \frac{1}{192} \sin(4y) \cos(4z) - \frac{1}{192} \sin(2y) \cos(4z) - \frac{1}{192} \sin(2y) \cos(4x) \\
&\quad + \frac{27}{3520} \cos(2z) \sin(4y) + \frac{1}{384} \cos(2x) \sin(4y) + \frac{43}{4224} \sin(2y) \cos(2z) + \frac{1}{96} \cos(2x) \sin(2y) \cos(4z) \\
W_3 &= -\frac{1}{384} \cos(2y) \cos(4x) \sin(4z) + \frac{19}{1584} \cos(2x) \cos(2y) \sin(2z) + \frac{79}{12672} \cos(4x) \cos(2y) \sin(2z) \\
&\quad + \frac{79}{12672} \cos(2x) \cos(4y) \sin(2z) + \frac{1}{192} \cos(4x) \sin(4z) - \frac{1}{384} \cos(2x) \sin(4z) - \frac{27}{1760} \sin(2z) \cos(4x) \\
&\quad - \frac{43}{4224} \sin(2z) \cos(2x) - \frac{1}{96} \cos(2x) \cos(2y) \sin(4z) - \frac{1}{384} \cos(2x) \cos(4y) \sin(4z) + \frac{1}{192} \cos(4y) \sin(4z) \\
&\quad + \frac{1}{384} \cos(2y) \sin(4z) - \frac{27}{1760} \cos(4y) \sin(2z) - \frac{43}{4224} \sin(2z) \cos(2y)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{P}{\rho}\right)_3 = & \frac{1}{640}\sin(x)\sin(3y)\sin(5z) - \frac{1}{640}\sin(3x)\sin(y)\sin(5z) + \frac{1}{31680}\sin(3x)\sin(5y)\sin(z) \\
& - \frac{1}{31680}\sin(5x)\sin(3y)\sin(z) - \frac{1}{544896}\sin(3x)\sin(5y)\sin(3z) + \frac{689}{300960}\sin(3x)\sin(y)\sin(3z) \\
& - \frac{689}{300960}\sin(x)\sin(3y)\sin(3z) + \frac{1}{544896}\sin(5x)\sin(3y)\sin(3z) + \frac{97}{246400}\sin(x)\sin(5y)\sin(3z) \\
& - \frac{1229}{190080}\sin(x)\sin(5y)\sin(z) - \frac{4337}{696960}\sin(x)\sin(3y)\sin(z) + \frac{1229}{190080}\sin(5x)\sin(y)\sin(z) \\
& - \frac{97}{246400}\sin(5x)\sin(y)\sin(3z) + \frac{4337}{696960}\sin(3x)\sin(y)\sin(z)
\end{aligned}$$

The mean rate of energy dissipation is:

$$\bar{W} = \mu(\overline{\xi^2} + \overline{\eta^2} + \overline{\zeta^2}) = \mu\overline{\omega^2},$$

where ξ , η and ζ are components of vorticity $\omega = \text{curl } V$, where V is the velocity vector and its x , y and z components are respectively u , v and w as defined previously. For enstrophy

$$\overline{\omega^2} = \frac{3}{4} + \frac{5t^2}{64} + \frac{25t^4}{4224} - \frac{52439t^6}{188334080} + \frac{729080006648669t^8}{11942691907829760000}$$

Symbolic Computations by Different Software

For symbolic computation of the solution of the Taylor-Green vortex, we developed three different codes for three different symbolic computation software, namely, Maple, Maxima and Mathematica. Our first attempt was a direct computation with Maple without any considerable optimization of computational routines. This code was very slow and consumed a large amount of memory. Later we developed a new code using Maxima. As it was mentioned by Fateman [11] for his own Macsyma code, using Poisson series and its related commands in Maxima was a very important factor for increasing speed of computations and decreasing memory consumption but it did not have a good performance at higher powers of time and the program crashed when number of terms and memory usage increased. We also developed a Mathematica code for solution of Taylor-Green vortex. This code showed the best performance between all of our codes did not have any of aforementioned problems of other software.

Conclusions

The time series expansion of the Euler equation for the unsteady three dimensional flow of an inviscid, incompressible, of Taylor' model, are obtained to 8 terms exactly by means of symbolic calculations and compared well with results of ref [11]. For this symbolic computations we used Maple, Maxima and Mathematica software and by comparing the performance of our codes in these three different software, Mathematica showed the best performance for symbolic computation of Taylor-Green vortex.

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