A general way to construct a new optimal scheme with eighth-order convergence for nonlinear equations

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Abstract

In this paper, we present a new and interesting optimal scheme of order eight in a general way for solving nonlinear equations, numerically. The construction of the scheme is based on rational function approach. The beauty of the proposed is that it is capable to produce further new and interesting optimal schemes of order eight from every existing optimal fourth-order scheme whose first substep employs Newton's method. The theoretical and computational properties of the proposed scheme are fully investigated along with main theorem which establishes the order of convergence and asymptotic error constant. Several numerical examples are given and analyzed in detail to demonstrate faster convergence and high computational efficiency of the proposed methods.

Keywords: Nonlinear equations, Simple roots, Computational order of convergence, Newton's method.

Introduction

With the advancement of digital computer, advanced computer arithmetics and symbolic computation, a special attention has been paid to the development of optimal eighth-order iterative methods in the past two decades. The merit of these methods is that they converge fast towards a sought root. Moreover, we can reach our desired accuracy in a very small number of iterations.

The researchers from the world wide proposed a large number of optimal eighth-order methods [3, 4, 7, 8, 9, 10, 11, 15, 16, 19, 20, 21, 23, 24]. Most of them are extensions of Newton's method or Newton-like method or any particular existing method like Ostrowski's method, King's method, King-type method, etc. at the expense of additional functional evaluations or increased number of substeps of the original methods. However, there still are a few number of optimal schemes which are applicable to every existing iterative method of particular order to further obtain higher-order methods.

In the recent years, Sharma et al. [20] have given an optimal eighth-order scheme in a general way, which is applicable to every optimal fourth-order method whose first substep is Newton's to further extend eighth-order convergence. But, it should be noted that they provided the third substep in their scheme without any justification. Optimal schemes applicable to any fourth-order iterative scheme with full justification of the development are more interesting and challenging task in the field of numerical analysis.

For the construction of a new iterative scheme, it is quite often used to approximate the functions or derivatives of the involved function. In the available literature, we have several kinds of approximations for e.g. Functional approach, Sampling approach, Geometric approach, Weight function approach, Adomain approach, Composition approach and Rational function approach.

Every approach has some advantages and disadvantages because it's dependent on the problem under consideration. The choice of suitable approximation approach not only produce simple and interesting schemes but also can save considerable amount of computation. Rational function approach is one of the most important techniques in numerical analysis for approximating the function or to find the next approximation.

In general, the number of tangency conditions are equal to number of undetermined constants. Further, we will get an improved method with higher-order convergence as we increase the number of undetermined constants in the rational function (for the details, see Jarratt and Nudds [26]).

The principle aim of this study is to present a new and interesting optimal scheme of order eight in general way instead of like earlier study [3, 4, 7, 8, 9, 10, 11, 15, 16, 19, 20, 21, 23, 24], where researchers proposed some eighth-order extensions of some particular methods like Ostrowski' method or King's family or Ostrowski-type, etc. Our proposed scheme is applicable on every existing optimal fourth-order scheme (which can be chosen from the available literature) whose first substep employs Newton's method to produce further new and interesting optimal eighth-order scheme. We construct this scheme with the help of rational approximation approach. In order to check the effectiveness and validity of our study, we compare them with the existing methods of same order on a concrete variety of nonlinear functions. From the numerical experiments, it is observed that our proposed methods perform better than existing ones.

Development of eighth-order optimal schemes

This section is devoted to the construction of an optimal and interesting eighth-order scheme in general way. Therefore, we consider a general fourth-order scheme whose first substep is classical Newton's method in the following way:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = \psi_4(x_n,\ y_n). \end{cases} \tag{1}$$

In order to obtain the next iteration and eighth-order convergence, we simply apply the classical Newton's method, which is given as follows:

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{2}$$

Since, the above scheme uses five functional evaluations. So, this scheme can not be optimal in the sense of Kung-Traub conjecture [15]. However, we can reduce the number of functional evaluations by introducing a rational function $\eta(x)$, which is defined as follows:

$$\eta(x) = \eta(x_n) + \frac{(x - x_n) + \alpha_1}{\alpha_2(x - x_n)^2 + \alpha_3(x - x_n) + \alpha_4}, \tag{3}$$

where $\alpha_i (1 \leq i \leq 4)$ are the disposable parameters. We will determine the values of these parameters with the help of following tangency conditions

$$\eta(x_n) = f(x_n),\ \eta'(x_n) = f'(x_n),\ \eta(y_n) = f(y_n),\ \eta(z_n) = f(z_n). \tag{4}$$

With the assumption of one more tangency condition $\eta'(z_n) = f'(z_n)$, we will obtain the last substep in the following way

$$x_{n+1} = z_n - \frac{f(z_n)}{\eta'(z_n)}, \tag{5}$$

which no longer requires the evaluation of $f'(z_n)$.

Now, with the help of first two tangency conditions, we will yield

$$\alpha_1 = 0,\ \alpha_4 = \frac{1}{f'(x_n)}. \tag{6}$$

Again, by using the last two tangency conditions and the above values of α_1 and α_4, we will obtain the following two linear independent equations

$$\begin{aligned}
\alpha_2(y_n - x_n) + \alpha_3 &= \frac{1}{(y_n - x_n)}\left[\frac{1}{f[y_n,\ x_n]} - \frac{1}{f'(x_n)}\right],\\
\alpha_2(z_n - x_n) + \alpha_3 &= \frac{1}{(z_n - x_n)}\left[\frac{1}{f[z_n,\ x_n]} - \frac{1}{f'(x_n)}\right],
\end{aligned} \tag{7}$$

which further yield

$$\begin{aligned}
\alpha_2 &= \frac{f(x_n)f[y_n,\ x_n]f[z_n,\ x_n] - f'(x_n)f[y_n,\ x_n](f(x_n) + f[z_n,\ x_n](x_n - z_n)) + f'(x_n)^2 f[z_n,\ x_n](x_n - z_n)}{f(x_n)f[y_n,\ x_n]f[z_n,\ x_n](x_n - z_n)(f(x_n) + f'(x_n)(z_n - x_n))},\\
\alpha_3 &= \frac{f'(x_n)\left(\frac{f(x_n)(f[z_n,\ x_n] - f'(x_n))}{f'(x_n)^2 f[z_n,\ x_n](x_n - z_n)} + \frac{(f'(x_n) - f[y_n,\ x_n])(x_n - z_n)}{f(x_n)f[y_n,\ x_n]}\right)}{f(x_n) + f'(x_n)(z_n - x_n)},
\end{aligned} \tag{8}$$

where $f[y_n,\ x_n] = \frac{f(y_n)-f(x_n)}{y_n - x_n}$ and $f[z_n,\ x_n] = \frac{f(z_n)-f(x_n)}{z_n - x_n}$ are divided difference of order one. With the help of expression (3), we can easily obtain

$$\eta'(z_n) = \frac{\alpha_4 - (z_n - x_n)^2\alpha_2}{\left[(z_n - x_n)^2\alpha_2 + (z_n - x_n)\alpha_3 + \alpha_4\right]^2}. \tag{9}$$

Finally, by using the expressions (1), (5) and (9), we obtain

$$\begin{cases}
y_n = x_n - \dfrac{f(x_n)}{f'(x_n)},\\
z_n = \psi_4(x_n,\ y_n),\\
x_{n+1} = z_n - \dfrac{f(z_n)\left[(z_n - x_n)^2\alpha_2 + (z_n - x_n)\alpha_3 + \alpha_4\right]^2}{\alpha_4 - (z_n - x_n)^2\alpha_2},
\end{cases} \tag{10}$$

where $\alpha_2,\ \alpha_3$ and a_4 are defined earlier in this section. The following Theorem 1 demonstrates three important things: first one is related to optimal eighth-order convergence without using any additional functional evaluations; second one is how a rational function $\eta(x)$ plays a vital role in the construction of iterative scheme (10) in a general way; third one is how a single coefficient B_1 in $\psi_4(x_n,\ y_n)$ contributes to its role in the construction of the desired asymptotic error constant.

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an interval containing ξ, where ξ is a simple zero of the involved function. In addition, we assume that $\psi_4(x_n,\ y_n)$ is any optimal fourth-order scheme whose first sub step employs Newton's method. Moreover, we* *consider initial guess $x = x_0$ is sufficiently close to ξ for guaranteed convergence. Then, the proposed scheme* (10) *has an optimal eighth-order convergence.*

Proof: Let us consider that $e_n = x_n - \xi$ be the error at nth term. The Taylor's series expansion of the function $f(x_n)$ and $f'(x_n)$ around $x = \xi$ with the assumption $f'(\xi) \neq 0$ leads us to:

$$f(x_n) = f'(\xi)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^7 + O(e_n^9)] \tag{11}$$

and

$$f'(x_n) = f'(\xi)[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+7c_7e_n^6+8c_8e_n^7+9c_9e_n^8+O(e_n^9)], \tag{12}$$

respectively, where $c_k = \frac{f^{(k)}(\xi)}{k!f'(\xi)}$ for $k = 2,\ 3,\ \ldots,\ 8$.
With the help of above expressions (11) and (12) in the first substep, we obtain

$$\begin{aligned}y_n - \xi = {}& c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (4c_2^3 - 7c_3c_2 + 3c_4)e_n^4 + (20c_3c_2^2 - 8c_2^4 - 10c_4c_2\\ & - 6c_3^2 + 4c_5)e_n^5 + \Big\{16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + (33c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6\Big\}e_n^6\\ & - 2\Big\{16c_2^6 - 64c_3c_2^4 + 36c_4c_2^3 + 9(7c_3^2 - 2c_5)c_2^2 + (8c_6 - 46c_3c_4)c_2 - 9c_3^3 + 6c_4^2\\ & + 11c_3c_5 - 3c_7\Big\}e_n^7 + \Big\{64c_2^7 - 304c_3c_2^5 + 176c_4c_2^4 + (408c_3^2 - 92c_5)c_2^3 + (44c_6\\ & - 348c_3c_4)c_2^2 + (118c_5c_3 - 135c_3^3 + 64c_4^2 - 19c_7)c_2 + 75c_3^2c_4 - 31c_4c_5 - 27c_3c_6\\ & + 7c_8\Big\}e_n^8 + O(e_n^9).\end{aligned} \tag{13}$$

Again, we obtain the following expansion of $f(y_n)$ about a point $x = \xi$ with the help of Taylor series

$$\begin{aligned}f(y_n) = f'(\xi)\Bigg[&c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 - 7c_3c_2 + 3c_4)e_n^4 - 2(6c_2^4 - 12c_3c_2^2\\ & + 5c_4c_2 + 3c_3^2 - 2c_5)e_n^5 + \{28c_2^5 - 73c_3c_2^3 + 34c_4c_2^2 + (37c_3^2 - 13c_5)c_2 + 5c_6\\ & - 17c_3c_4\}e_n^6 - 2\{32c_2^6 - 103c_3c_2^4 + 52c_4c_2^3 + (80c_3^2 - 22c_5)c_2^2 + (8c_6 - 52c_3c_4)c_2\\ & - 9c_3^3 + 6c_4^2 + 11c_3c_5 - 3c_7\}e_n^7 + O(e_n^8)\Bigg].\end{aligned} \tag{14}$$

By using the expression (11), (13) and (14), we have

$$\begin{aligned}\frac{f(y_n) - f(x_n)}{y_n - x_n} = {}& 1 + c_2e_n + (c_2^2 + c_3)e_n^2 + (3c_3c_2 - 2c_2^3 + c_4)e_n^3 + (4c_2^4 - 8c_3c_2^2 + 4c_4c_2\\ & + 2c_3^2 + c_5)e_n^4 + \{20c_3c_2^3 - 8c_2^5 - 11c_4c_2^2 + (5c_5 - 9c_3^2)c_2 + 5c_3c_4 + c_6\}e_n^5\\ & + \{16c_2^6 - 48c_3c_2^4 + 29c_4c_2^3 + (31c_3^2 - 14c_5)c_2^2 + 6(c_6 - 4c_3c_4)c_2 - 2c_3^3\\ & + 3c_4^2 + 6c_3c_5 + c_7\}e_n^6 + O(e_n^7).\end{aligned} \tag{15}$$

Since, $\psi_4(x_n,\ y_n)$ is an optimal fourth-order scheme. So, it will satisfy the error equation of the following form

$$z_n - \xi = B_1e_n^4 + B_2e_n^5 + B_3e_n^6 + B_4e_n^7 + B_5e_n^8 + O(e_n^9), \tag{16}$$

where $B_1 \neq 0$.

Now, we can expand the function $f(z_n)$ about a point $z = \xi$ with the help of Taylor series expansion, which is given as follows

$$f(z_n) = f'(\xi)\left[B_1 e_n^4 + B_2 e_n^5 + B_3 e_n^6 + B_4 e_n^7 + (B_1^2 c_2 + B_5)e_n^8 + O(e_n^9)\right]. \tag{17}$$

By using the expression (11), (16) and (17), we obtain

$$\begin{aligned}\frac{f(z_n) - f(x_n)}{z_n - x_n} &= 1 + c_2 e_n + c_3 e_n^2 + c_4 e_n^3 + (B_1 c_2 + c_5)e_n^4 + (B_2 c_2 + B_1 c_3 + c_6)e_n^5 \\ &+ (B_3 c_2 + B_2 c_3 + B_1 c_4 + c_7)e_n^6 + (B_4 c_2 + B_3 c_3 + B_2 c_4 + B_1 c_5 \\ &+ c_8)e_n^7 + O(e_n^8).\end{aligned} \tag{18}$$

Now, with the help of expressions (11) – (18), we further obtain

$$\begin{aligned}\frac{f(z_n)\left[(z_n - x_n)^2\alpha_2 + (z_n - x_n)\alpha_3 + \alpha_4\right]^2}{\alpha_4 - (z_n - x_n)^2\alpha_2} &= B_1 e_n^4 + B_2 e_n^5 + B_3 e_n^6 + B_4 e_n^7 \\ &- B_1 c_2(B_1 + c_2^3 - 2c_2 c_3 + c_4)e_n^8 + O(e_n^9).\end{aligned} \tag{19}$$

Finally, by inserting the expressions (16) and (19) in the last substep of the proposed scheme (10) and after some simplification, we obtain

$$e_{n+1} = B_1 c_2(B_1 + c_2^3 - 2c_2 c_3 + c_4)e_n^8 + O(e_n^9), \tag{20}$$

This completes the proof. □

Remark 2 *The above asymptotic error constant* (20) *reveals that the proposed scheme* (10) *attains an optimal eighth-order convergence in the sense of Kung-Traub conjecture. In addition, one generally expects that the asymptotic error constant of the proposed scheme* (10) *also contains some constants namely,* $c_2, c_3, c_4, c_5, c_6, c_7, c_8$ *and* B_1, B_2, B_3, B_4, B_5*. However, only* $B_1,\ c_2,\ c_3$ *and* c_4 *appears in the asymptotic error constant which can be seen in* (20)*. This simplicity clearly reflects that our current rational function approach with the tangency conditions which is used for the reduction of functional evaluations, plays a vital role in the development of an optimal eighth-order method.*

Numerical experiments

In this section, we will check the effectiveness and validity of our theoretical results which we have proposed in Section 2. For this purpose, we shall consider a concrete variety of nonlinear equations, which are given as follows:

$f_1(x) = e^x \sin(x) + \log(x^2 + 1)$;[5]	$\xi = 0$
$f_2(x) = x^6 - x^4 - x^3 - 1$; [23]	$\xi = 1.4036021248742166432791385576$8
$f_3(x) = e^x - 4x^2$; [16]	$\xi = 0.714805912362777806137622208112$
$f_4(x) = \tan^{-1}(x) - x + 1$; [1]	$\xi = 2.13226772527288513162542069694$
$f_5(x) = e^{-x} + \cos(x)$; [19]	$\xi = 1.74613953040801241765070308895$
$f_6(x) = \log x$;[17]	$\xi = 1$

First of all, we shall verify the theoretical order of convergence of the proposed methods on the basis of the results obtained from $\left|\frac{x_{n+1} - x_n}{(x_n - x_{n-1})^8}\right|$ and computational order of convergence. In Table

1, we displayed the number of iteration indexes (n), approximated zeros (x_n), absolute residual error of the corresponding function $(|f(x_n)|)$, error in the consecutive iterations $|x_{n+1} - x_n|$, $\left|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}\right|$, the asymptotic error constant $\eta = \lim\limits_{n\to\infty}\left|\dfrac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}\right|$ and computational order of convergence (ρ). In order to calculate the computational order of convergence (ρ), we use the following formula:

$$\rho = \left|\frac{(x_{n+1}-x_n)/\eta}{(x_n - x_{n-1})}\right|, \; n = 1, 2, 3.$$

We calculate the computational order of convergence, asymptotic error constant and other constants up to several number of significant digits (minimum 1000 significant digits) to minimize the round off error. But due to the limited paper space, we display the value of x_n and ρ up to 15 and 6 significant digits, respectively. In addition, we also display $\left|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}\right|$ and η up to 10 significant digits. Moreover, absolute residual error in the function $|f(x_n)|$ and error in the consecutive iterations $|x_{n+1} - x_n|$ are displayed up to 2 significant digits with/without exponent power which are mentioned in Tables 1. Furthermore, the approximated zeros up to 30 significant digits are also displayed in Table 1 although minimum 1000 significant digits are available with us.

For the computer programming, all computations have been performed using the programming package $Mathematica$ 9 with multiple precision arithmetic. Further, the meaning of $a(-b)$ is $a \times 10^{(-b)}$ in the following Tables 1.

Now, we want to see the comparison of our methods with the other existing optimal methods of same order. Therefore, we consider some special cases of the proposed scheme in the following way

(i) Let us consider the well known fourth-order King's family [13]. By using King's family in the proposed scheme, we obtain a new optimal eighth-order extension of King's family, which is defined as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \left[\dfrac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)}\right] \dfrac{f(y_n)}{f'(x_n)}, \; \beta \in \mathbb{R}, \\ x_{n+1} = z_n - \dfrac{f(z_n)\left[(z_n - x_n)^2\alpha_2 + (z_n - x_n)\alpha_3 + \alpha_4\right]^2}{\alpha_4 - (z_n - x_n)^2\alpha_2}. \end{cases} \tag{21}$$

For a computational point of view, let us consider $\beta = 0$ in the above scheme, called by $(OM1)$.

(ii) Now, we shall choose another optimal family of fourth-order methods proposed by Chun in [5]. Then, we obtain another new optimal family of eighth-order methods, which is described as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \left[\dfrac{\{f(x_n)\}^2}{\{f(x_n)\}^2 - 2f(x_n)f(y_n) + 2\beta\{f(y_n)\}^2}\right]\dfrac{f(y_n)}{f'(x_n)}, \quad \beta \in \mathbb{R}, \\ x_{n+1} = z_n - \dfrac{f(z_n)\left[(z_n - x_n)^2\alpha_2 + (z_n - x_n)\alpha_3 + \alpha_4\right]^2}{\alpha_4 - (z_n - x_n)^2\alpha_2}. \end{cases} \tag{22}$$

Let us choose $\beta = \frac{1}{4}$ in the above scheme for computational experiments, known by $(OM2)$.

(iii) Again, we consider another optimal family of fourth-order methods proposed by Behl et al. [1]. With the help of our proposed scheme (9), we obtain the following optimal family of eighth-order methods

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{f(x_n)}{f'(x_n)}\left[\dfrac{(b_1^2 + b_1b_2 - b_2^2)f(x_n)f(y_n) - b_1(b_1 - b_2)\{f(x_n)\}^2}{\Big(b_1f(x_n) - b_2f(y_n)\Big)\Big((2b_1 - b_2)f(y_n) - (b_1 - b_2)f(x_n)\Big)}\right], \\ x_{n+1} = z_n - \dfrac{f(z_n)\left[(z_n - x_n)^2\alpha_2 + (z_n - x_n)\alpha_3 + \alpha_4\right]^2}{\alpha_4 - (z_n - x_n)^2\alpha_2}, \end{cases} \tag{23}$$

where $b_1,\ b_2 \in \mathbb{R}$ such that $b_1 \neq 0$ & b_2. For a computational point of view, let us consider $b_1 = 1$ and $b_2 = \frac{1}{10}$ in the above scheme, denoted by $(OM3)$.

In the similar way, we can choose any optimal fourth-order iterative method/family of iterative methods from available literature whose first substep employs Newton's method to further obtain optimal eighth-order iterative method/family of iterative methods.

Now, we will compare them on a concrete variety of nonlinear functions with the following optimal eighth-order methods

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \left[\dfrac{f(x_n)}{f(x_n) - 2f(y_n)}\right]\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n + \dfrac{f(x_n)f(z_n)\left(f(x_n) + 2f(z_n)\right)\left(f(y_n) + f(z_n)\right)}{f'(x_n)f(y_n)\left(2f(x_n)f(y_n) - f(x_n)^2 + f(y_n)^2\right)}, \end{cases} \tag{24}$$

$$
\begin{cases}
u_n = x_n + \alpha f(x_n),\ \alpha \in \mathbb{R}, \\
y_n = x_n - \dfrac{\alpha f(x_n) f(x_n)}{f(u_n) - f(x_n)}, \\
z_n = = y_n - \dfrac{f(y_n)}{-\frac{f(u_n)(x_n-y_n)}{\alpha f(x_n)(\alpha f(x_n)+x_n-y_n)} + \frac{\alpha f(x_n)+x_n-y_n}{\alpha(x_n-y_n)} - \frac{f(y_n)(\alpha f(x_n)+2x_n-2y_n)}{(x_n-y_n)(\alpha f(x_n)+x_n-y_n)}}, \\
x_{n+1} = z_n - \dfrac{f(z_n)(u_n-x_n)(u_n-y_n)(u_n-z_n)(x_n-y_n)(x_n-z_n)(y_n-z_n)}{a_1 - a_2 f(z_n)(u_n-x_n)(u_n-y_n)(x_n-y_n)},
\end{cases} \tag{25}
$$

where $a_1 = f(y_n)(u_n - x_n)(u_n - z_n)^2(x_n - z_n)^2 + f(y_n)(u_n - x_n)(u_n - z_n)^2(x_n - z_n)^2 + (y_n - z_n)^2(f(u_n)(x_n - y_n)(x_n - z_n)^2 - f(x_n)(u_n - y_n)(u_n - z_n)^2),\ a_2 = (u_n(x_n + y_n - 2z_n) + x_n(y_n - 2z_n) + z_n(3z_n - 2y_n)),$

$$
\begin{cases}
y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\
z_n = y_n - \left[\dfrac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\right] \dfrac{f(y_n)}{f'(x_n)}, \\
x_{n+1} = z_n - \dfrac{f(z_n)}{2f[z_n,\ x_n] - f'(x_n)} \left[1 + \dfrac{f(z_n)}{f(y_n)} + \left(\dfrac{f(y_n)}{f'(x_n)}\right)^3 - \dfrac{2f(z_n)}{f'(x_n)} \right. \\
\qquad \left. - \dfrac{31}{4}\left(\dfrac{f(y_n)}{f(x_n)}\right)^4 - \dfrac{3}{2}\left(\dfrac{f(y_n)}{f(x_n)}\right)^3 + \left(\dfrac{f(z_n)}{f(x_n)}\right)^2 + \left(\dfrac{f(z_n)}{f(y_n)}\right)^2\right],
\end{cases} \tag{26}
$$

$$
\begin{cases}
w_n = x_n + \beta f(x_n),\ \beta \in \mathbb{R} \\
y_n = x_n - \dfrac{\beta f(x_n) f(x_n)}{f(w_n) - f(x_n)}, \\
z_n = y_n - \dfrac{f(w_n) f(y_n)(y_n - x_n)}{(f(w_n) - f(y_n))(f(y_n) - f(x_n))}, \\
x_{n+1} = z_n - \dfrac{f(w_n) f(y_n)\left(\frac{f(x_n)(z_n - x_n)}{f(z_n) - f(x_n)} - x_n + y_n\right)}{(f(w_n) - f(z_n))(f(y_n) - f(z_n))} + \dfrac{f(y_n)(z_n - y_n)}{f(z_n) - f(y_n)},
\end{cases} \tag{27}
$$

$$
\begin{cases}
y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\
z_n = x_n - \left[\dfrac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\right] \dfrac{f(x_n)}{f'(x_n)}, \\
u_n = z_n - \left(\dfrac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} + \dfrac{f(z_n)}{2(f(y_n) - 2f(z_n))}\right)^2 \dfrac{f(z_n)}{f'(x_n)}, \\
x_{n+1} = u_n - \dfrac{3(b_2 + b_3) f(z_n)(u_n - z_n)}{f'(x_n)(b_1(u_n - z_n) + b_2(y_n - x_n) + b_3(z_n - x_n))}
\end{cases} \tag{28}
$$

and

$$\begin{cases} y_n =x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n =y_n - \left[\dfrac{f(y_n)}{f(x_n)-2f\,(y_n)}\right]\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} =z_n - \left[\dfrac{6f(y_n)^4\{f(x_n)+5f(y_n)\}}{f(x_n)^5}\right]\dfrac{f(z_n)}{f'(x_n)} \\ \qquad - \dfrac{f(x_n)+31f(z_n)}{f(x_n)+30f(z_n)}\left[\dfrac{f[y_n,\ x_n]f(z_n)}{f[z_n,\ x_n]f[y_n,\ z_n]}\right], \end{cases} \tag{29}$$

which were proposed by Džunić and Petković [9], Khattri and Steihaug [11] (for $\alpha = 1$), Soleymani et al. in [21], Kung and Traub [15] (for $\beta = 1$), Cordero et al. in [7] (for $b_1 = 1,\ b_2 = 1,\ b_3 = 2$) and Heydari et al. [10], respectively called by $DP,\ KS,\ SM,\ KT,\ CM,$ and HM.

For comparisons of our proposed methods with the other existing ones, we experimented with the functions $f_i(x),\ i = 1,..,6$. We have taken 500 equally spaced points $\{t_i\}_{i=0}^{500}$ in the interval $[-3,3]$ for $f_i(x),\ i = 1,..,5$ and in $[0.1,6.1]$ for $f_6(x)$ as initial points for the methods. Notice that $f_2(x) = 0$ contains two solutions $\xi = 1.40360212487421664327913855768, \xi = -1$ in $[-3,\ 3]$, and the others only one solution.

If x_0 attempts a root with tolerance $\epsilon = 10^{-5}$ in 14 iterations we have decided it converged to the root, otherwise, it diverged. We have registered the total number of iterations required to converge to a root and also collected the CPU time in seconds required to run each method on all the points using Samsung desktop computer with Intel(R) Core(TM) i5-4590 CPU. We then computed the average number of iterations required per point and the number of points requiring more than 14 iterations.

We have averaged performance results for the methods in comparison in Tables 2-4 across the 6 test functions. Based on Table 2 we find that the minimum the number of divergent points on average is achieved by OM1 (5.67 out of 500 points) followed by KS (6 points), OM3 (11.3 points), DP (37.8 points) and OM2 (67.5 points). All the others have $150 - 286.3$ number of points requiring more than 14 iterations on average. We will remove these methods from further consideration, since these methods have more than 24 percent of divergence. In terms of CPU time (see Table 3), the fastest method is DP (0.973 seconds) closely followed by OM1 (1.158 seconds), KS (1.287 seconds) and OM3 (1.3 seconds). The slowest is OM2 (2.231 seconds), which will be removed from further discussion. Recall that although SM is the fastest of all the methods considered, it is no longer being considered now since it is one of the methods having more than 24 percent of initial points diverged. Consulting the average number of iterations per point on average (see Table 4), we find that OM1 is best (2.49) followed by KS (2.58) and OM3 (2.63). The worst is DP (3.30).

In view of our analysis of the results in Tables 2-4 given above, the best method overall is OM1.

Conclusions

In this paper we proposed a new optimal eighth-order family of methods based on rational function in a general way. Some of our methods have been compared to several existing methods of the same order. OM1, one of our methods, is found to be the best method based on 3 quantitative criteria (Divergence percent, CPU time, Average number of iterations per point), confirming that the proposed methods are highly efficient as compared to the existing methods.

Table 1: Convergence behavior of methods $OM1,\ OM2$ and $OM3$ on each test function

Cases	$f(x)$	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}$	η	ρ
$OM1$	f_1	0	0.5	1.0	5.0(−1)			
		1	0.00306695875782981	3.1(−3)	3.1(−3)	8.247549737(−1)	1.980000000(+2)	15.8377
		2	1.48036410450262(−18)	5.1(−18)	5.1(−18)	1.891058911(+2)		8.00794
		3	4.56681645644905(−141)	4.6(−141)	4.6(−141)	1.980000000(+2)		8.00000
$OM1$	f_2	0	1.5	2.0	9.6(−2)			
		1	1.40360330825001	1.9(−5)	1.2(−6)	1.587178031(+2)	4.605587105(+2)	8.45540
		2	1.40360212487422	2.8(−44)	1.8(−45)	4.605524658(+2)		8.00000
		3	1.40360212487422	7.0(−355)	4.5(−356)	4.605587105(+2)		8.00000
$OM2$	f_3	0	0.6	3.8(−1)	1.1(−1)			
		1	0.714806004989988	3.4(−7)	9.3(−8)	3.069175663	1.085366264	7.51976
		2	0.714805912362778	2.2(−56)	5.9(−57)	1.085365407		8.00000
		3	0.714805912362778	5.7(−450)	1.6(−450)	1.085366264		8.00000
$OM2$	f_4	0	2.4	2.2(−1)	2.7(−1)			
		1	2.13226772533188	4.8(−11)	5.9(−11)	2.234686093(−6)	5.519129858(−6)	8.68610
		2	2.13226772527289	6.6(−88)	8.1(−88)	5.519129857(−6)		8.00000
		3	2.13226772527289	8.4(−703)	1.1(−702)	5.519129858(−6)		8.00000
$OM3$	f_5	0	1.5	2.9(−1)	2.5(−1)			
		1	1.74613952980597	7.0(−10)	6.0(−10)	4.468629204(−5)	1.786446252(−4)	8.98850
		2	1.74613953040801	3.6(−78)	3.1(−78)	1.786446246(−4)		8.00000
		3	1.74613953040801	1.7(−624)	1.5(−624)	1.786446252(−4)		8.00000
$OM3$	f_6	0	0.5	6.9(−1)	5.0(−1)			
		1	0.999983241870036	1.7(−5)	1.7(−5)	4.291231744(−3)	8.979552469(−4)	5.74343
		2	1.00000000000000	5.6(−42)	5.6(−42)	8.979882433(−4)		8.00000
		3	1.00000000000000	8.5(−334)	8.5(−334)	8.979552469(−4)		8.00000

(It is straightforward to say that our methods not only converging very fast to the desired zero but also have the smaller asymptotic error constant which confirm the theoretical results.)

Table 2: Number of points requiring more than 14 iterations for each test function (1–6) to corresponding method and divergence percentage

Methods	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	average	Divergence Percentage
OM1	1	7	1	19	6	0	5.67	1.13%
OM2	17	26	12	27	21	302	67.5	13.5%
OM3	2	17	1	12	36	0	11.3	2.26%
DP	7	32	2	179	6	1	37.8	7.56%
KS	19	9	2	0	6	0	6	1.2%
SM	83	136	37	180	118	346	150	30%
KT	77	500	500	0	24	500	266.8	53.4%
CM	123	500	332	190	500	73	286.3	57.3%
HM	72	137	33	134	38	330	124	24.8%

Table 3: CPU time (in seconds) required for each test function (1–6) to corresponding method

Methods	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	average
OM1	2.153	0.578	0.593	1.077	1.280	1.264	1.158
OM2	4.228	1.388	1.139	1.856	1.950	2.824	2.231
OM3	2.215	0.686	0.671	1.108	1.607	1.513	1.3
DP	1.825	0.390	0.437	0.702	1.155	1.326	0.973
KS	2.356	0.656	0.687	1.014	0.827	2.184	1.287
SM	1.732	0.374	0.515	0.749	0.936	0.764	0.845
KT	4.337	2.215	3.183	3.447	2.683	3.479	3.224
CM	1.981	0.437	1.217	1.779	1.248	2.714	1.563
HM	1.810	0.406	0.499	0.827	1.092	0.670	0.884

Table 4: Average number of iterations per point for each test function (1–6) to corresponding method

Methods	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	average
OM1	2.43	3.10	2.45	2.72	2.51	1.71	2.49
OM2	4.92	6.34	4.90	4.47	3.90	9.88	5.74
OM3	2.42	3.42	2.44	2.48	3.16	1.88	2.63
DP	2.61	3.73	2.48	6.23	2.55	2.18	3.30
KS	3.04	3.97	2.74	1.64	1.66	2.42	2.58
SM	4.25	5.75	3.22	6.30	4.87	10.23	5.77
KT	6.42	14	14	6.37	5.88	14	10.11
CM	5.52	14	13.41	9.19	14	5.44	10.26
HM	4.01	5.82	3.24	5.35	3.02	9.80	5.21

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