Auto-adaptive resolution strategy for wave equations in integral formulation

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Abstract

This work develops and analyzes an auto-adaptive resolution strategy applied to wave equations solved by a Boundary Element Method (BEM). The proposed method is built using two essential tools: a localized *a posteriori* error estimate and a H-matrix compression technique. The first one guides a local mesh refinement strategy in order to reduce the error at each iteration of the loop. The second one calculates an approximation of the Galerkin matrix and thus enables the method to solve problems of high dimension. We show that the H-matrix is very efficient to deal with the very refined and heterogeneous meshes obtained from the auto-adaptive algorithm. Numerical experiments in electromagnetism tend to prove the efficiency of this resolution strategy. It tends to be a simple to use, reliable and computationally "affordable" method.

Keywords: **BEM, integral equation, wave equation, *a posteriori* error estimate, H-matrix, auto-adaptive strategy, EFIE.**

Introduction

Integral formulations are known to be very efficient at solving wave propagation problems in homogeneous medias. They are used daily in various engineering fields such as antenna design, RADAR signature computations or in buildings' acoustic studies. In all these contexts, there is a growing need for the control of the quality of the numerical solution. In standard Finite Element Method, this is usually done by using a *posteriori* error estimates to locally refine the mesh. We propose here an adaptation of this strategy to integral formulations. Two major difficulties arise in this case.

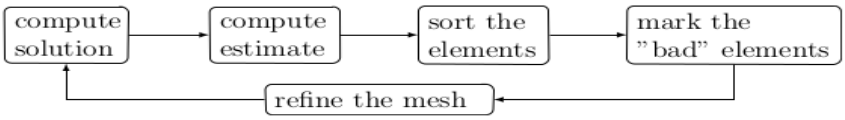
First, the sought solution lives in fractional order Sobolev spaces with non local norm [1]. The mathematical setting is thus trickier than in standard finite element method. It is possible to build an a *posteriori* error estimate, for instance based on the (classical) residual function. However the non local nature of both the fractional norm and the underlying operator make the localization of the error on the mesh very difficult.

The second main difficulty comes from the fact that the Galerkin matrix obtained from the BEM method is dense. This is an other consequence of the non locality of the integral

operator. Thus its storage cost varies in $O(n^2)$, with n being the size of the problem. This quadratic evolution of the memory cost makes the use of a matrix compression technique essential.

Proposed auto-adaptive procedure

We remind to the reader the global architecture of an auto-adaptive algorithm:



We propose the two following methods in order to overcome the difficulties exposed previously.

First, we use the localization of the residual function obtained by [2]. Their work gives a reliable indicator of the error on the mesh. Then [3] proved that an auto-adaptive loop built with this indicator combined with a Dörfler selection technique [4] and a newest vertex bisection refinement algorithm [5] converges to the exact solution. Howeover, one shall keep in mind that the practical computation of this estimator requires a real development effort.

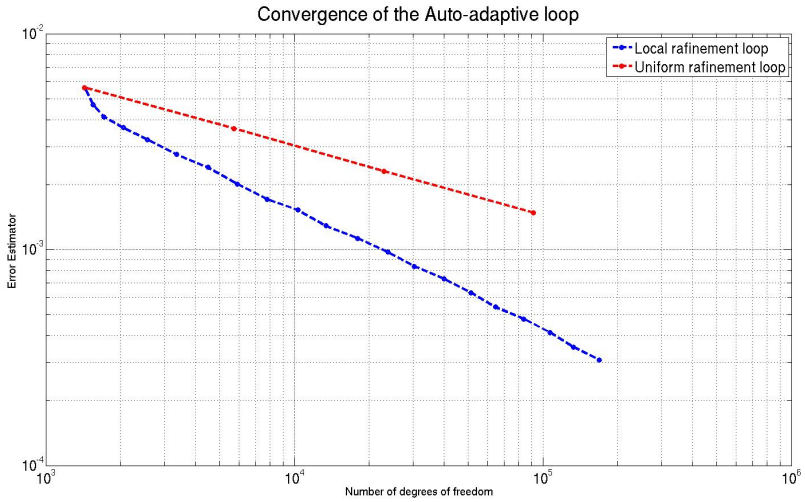
Secondly, we choose here to use a H-matrix methodology [6] in order to compress the Galerkin matrix. This family of algorithm is expected to be efficient and robust regarding the great heterogeneity of the meshes obtained by local refinement. We numerically prove its efficiency in an auto-adaptive context.

Numerical results

We validate the proposed auto-adaptive architecture with the problem of a perfectly conducting cube of size 1 meter excited by a plane wave of wave number 5. Here we use an Electrical Field Integral Equation (EFIE) formulation. Despite the simplicity of the shape, The geometrical singularities (the corners) make it a hard case.

Convergence of the error estimate

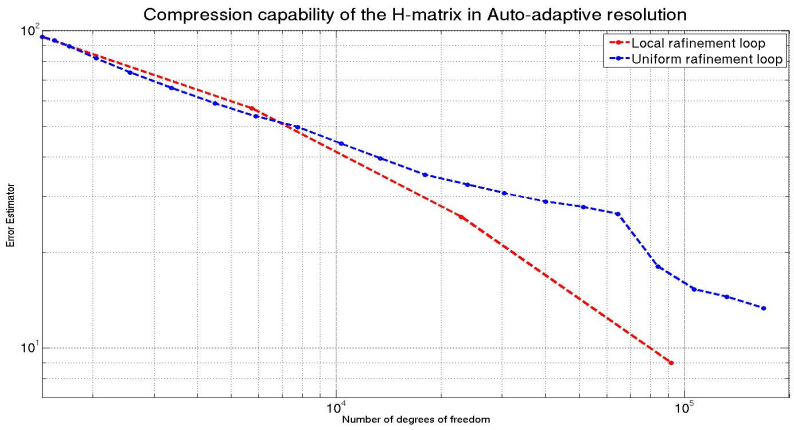
We show below the evolution of the value error estimate with respect to the number of degrees of freedom. The comparison with the uniform refinement proves the pertinence of this approach. One shall note that the error's indicator is proportionnal to the actual error.



The auto-adaptive procedure yields a convergence rate of $O(N^{-0,65})$, whereas the uniform refinement one is of $O(N^{-0,32})$ (N being the number of degrees of freedom). For the same accuracy prescrition, the uniform refinement leads to a much bigger mesh than the one obtained with the auto-adaptive algorithm. Howeover the latter needs more iterations to converge. One sees that the key to obtain a computationally competitive method is to reduce the cost of each iteration. One way do so would be to save the parts of the Galerkin matrix which are not modified by the local refinement of the mesh.

Efficiency of the compression technique

We show here that the H-matrix compression used is well adapted to the auto-adaptive context. We use the compression rate to measure the quality of the compression. It represents the memory cost of the Galerkin matrix expressed in percentage of the memory cost of the non compressed matrix. We obtain the following results:



The heterogeneity of the locally refined mesh leads to lower performances of the H-matrix. Howeover they keep being acceptable as we get a compression rate of roughly 10 % for the last iteration. This proves the robusteness of the method in an auto-adaptive context.

This compression methods also enables a quick computation of the H-matrix product operation. This property is very relevant in a context where an iterative resolution of the linear system is required.

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