An optimal reconstruction of Chebyshev-Halley type methods with local convergence analysis

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Abstract

In this manuscript, our principle aim is to present a new reconstruction of classical Chebyshev-Halley scheme having optimal fourth and eighth-order convergence for all α unlike the earlier studies. In addition, we analyze the local convergence of them by using supposition requiring the first-order derivative of the involved function f and the Lipschitz conditions. The new approach is not only the extension of earlier studies, but also formulates their theoretical radius of convergence. Several numerical examples originated from real life problems demonstrate that they are applicable to a broad range of scalar equations where previous studies cannot be used. Finally, dynamic study of them also demonstrates that bigger and promising basins of attractions belongs to our iteration functions.

Keywords: Nonlinear equations, Newton's method, Complex dynamics, Chebyshev-Halley method, local convergence analysis.

Introduction

Among the most harder and earlier issues of computational methods and numerical analysis are concerning with the cost-effective and accurate simple zeros of function $f(x)$ in a small number of iterations with specific degree of accuracy (where $f : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ is a univariate sufficiently smooth function in the closed interval $\mathbb{D}$). It is hard to find analytical methods in the available literature for solving such type of problems. So, there is only one option left for us to find the approximate solutions by using iterative procedures. One of the best and most famous iterative procedure is the classical Newton's method [30, 25]. Several higher-order reconstruction of Newtons procedure have been constructed at the expense of some other values of function/s and/or its derivative/s. We have a good number of cubically convergent iterative procedures, (for the details please see [30]) and one of them is given as below:

$$x_{n+1} = x_n - \left[1 + \frac{1}{2}\frac{L_f(x_n)}{1 - \alpha L_f(x_n)}\right]\frac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R}, \tag{1}$$

where $L_f(x_n) = \frac{f''(x_n)f(x_n)}{\{f'(x_n)\}^2}$. This is a well-known family of Chebyshev-Halley iteration functions [14]. We can easily obtain some popular iteration functions from this family. For example, the classical Chebyshev's method [30, 15], Halley's method [30, 15] and super-Halley method [30, 15] if we choose $\alpha = 0,\ \alpha = \frac{1}{2}$ and $\alpha = 1$, respectively. Regardless of cubic convergence, the scheme (1) is consider less practical from a computational point of view because it is not an easy task to find the second-order derivative of every problem.

This fact has motivated many scholars to turn towards the approach of multi-point iteration functions. The principal objective of them is to produce second or higher-order derivative free iteration functions with maximum convergence order by using certain values of function/s and or its first-order derivative/s. In 1964, Traub [30] presented the analysis of multi-point iteration functions with their properties. Recently, Petković et al. [25] also revised and update the facts about them.

Despite of going in to the detail of them, we have only focused on the higher-order and second derivative free modifications of the family (1). According to our expertise, many researchers from worldwide like, Kou and Li [18, 19], Kou [17], Chun [9], Amat et al. [1], Xiaojian [32] and Arygros et al. [5], proposed higher-order modification of Chebyshev-Halley's iteration functions not using the values of second or higher-order derivative/s.

Recently, Li et al. [21], presented an improvement of Chebyshev-Halley iteration functions, which is defined as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \left(1 + \dfrac{f(y_n)}{f(x_n) - 2\alpha f(y_n)}\right) \dfrac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R} \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(x_n) + f''(x_n)(z_n - x_n)}, \end{cases} \tag{2}$$

where $f''(x_n) = \frac{2f(y_n)f'(x_n)^2}{f(x_n)^2}$. The above scheme has minimum fifth-order convergence and further reaches at six for $\alpha = 1$.

Moving ahead in this direction, Sharma [29], also constructed the following new modification of the above scheme (2):

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \left(1 + \dfrac{f(y_n)}{f(x_n) - 2\alpha f(y_n)}\right) \dfrac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R} \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[z_n,\ y_n] + f[z_n,\ y_n,\ x_n](z_n - y_n) + f[z_n,\ y_n, x_n,\ x_n](z_n - y_n)(z_n - x_n)}. \end{cases} \tag{3}$$

This family (3) has minimum sixth-order convergence and further attains eighth for $\alpha = 1$. So, it means that this scheme has an optimal eighth-order convergence but only for $\alpha = 1$.

Both of the above mentioned schemes namely, (2) and (3), using three values of the considered function and one derivative of first-order at per step. But, none of them achieved an optimal convergence for each α. No doubts, Sharma got a little success in this path but that one is valid only for $\alpha = 1$, not for other values. According to Kung-Traub conjecture, we can attain maximum eighth-order convergence by using the same functional evaluations.

While keep all these things in our mind, we intend to propose a new powerful and an optimal reconstruction of Chebyshev-Halley iteration functions of order four and eight. In addition, we analyze the local convergence of them by using suppositions requiring first-order derivative of the involved function f and the Lipschitz conditions. Moreover, we also present their theoretical radius of convergence which provides guaranteed convergence of them. Further, we will give a practical exhibition of our iteration functions to many real life situations and conclude that they perform better than the earlier studies. Finally, the dynamical behavior of them also illustrate the above consequences to a great extent.

Construction of higher-order optimal schemes

First of all in this section, we propose a new reconstruction of fourth-order Chebyshev-Halley methods, not requiring the computation of second-order derivative. Then, we extend the same scheme for eighth-order convergence. For this purpose, we consider the well-known second order Newton's method [30], which is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{4}$$

With the help of Taylor series, we will get the following expansions of functions $f(y_n)$ and$f'(y_n)$ about a point $x = x_n$, as follow

$$f(y_n) \approx \frac{(y_n - x_n)^2}{2} f''(x_n), \tag{5}$$

and

$$f'(y_n) \approx f'(x_n) + f''(x_n)(y_n - x_n). \tag{6}$$

By adding the expressions (5) and (6), we get

$$f(y_n) + f'(y_n) \approx \frac{(y_n - x_n)^2}{2} f''(x_n) + f'(x_n) + f''(x_n)(y_n - x_n), \tag{7}$$

which further yields

$$f''(x_n) \approx \frac{2[f'(x_n)]^2\Big(f'(x_n) - f'(y_n) - f(y_n)\Big)}{f(x_n)\Big(2f'(x_n) - f(x_n)\Big)}. \tag{8}$$

However, this new approximation for $f''(x_n)$ uses four functional evaluations, viz. $f(x_n)$,$f'(x_n)$, $f(y_n)$, $f'(y_n)$. Therefore, in order to reduce the number of functional evaluations, we consider an approximation similar to the King's approximation [16], which is defined as follows

$$f'(y_n) = f'(x_n)\left(\frac{1 + \beta_1 v}{1 + \beta_2 v}\right), \tag{9}$$

where $v = \frac{f(y_n)}{f(x_n)}$ and $\beta_1, \beta_2 \in \mathbb{R}$.
Now, using the above expressions (8) and (9) in expression (1), we obtain a new reconstruction of Chebyshev-Halley family

$$x_{n+1} = x_n - \left[1 + \frac{1}{2}\frac{L_f^*(x_n)}{1 - \alpha L_f^*(x_n)}\right]\frac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R}, \tag{10}$$

where $L_f^*(x_n) = \dfrac{2f(y_n)[(\beta_1 - \beta_2)u^{-1} + \beta_2 v + 1]}{\Big(f(x_n) - 2f'(x_n)\Big)\Big(1 + \beta_2 v\Big)}$.

In order to attain eighth-order convergence of the scheme (10), we rewrite in the following way:

$$\begin{aligned} y_n =& x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n =& x_n - \left[1 + \frac{1}{2}\frac{L_f^*(x_n)}{1-\alpha L_f^*(x_n)}\right]\frac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R}, \\ x_{n+1} =& x_n - \frac{u\Big(f(x_n)-f(z_n)\Big)f(y_n) + (x_n - z_n)(v-1)f(x_n)f(z_n)}{(2v-1)(x_n-z_n)f'(x_n)f(z_n) + \Big(uf[x_n,z_n]-f(z_n)\Big)f(y_n)}, \end{aligned} \tag{11}$$

where $u = \frac{f(x_n)}{f'(x_n)}$ and $v = \frac{f(y_n)}{f(x_n)}$. The next theorem 1 indicates that under what choices on disposable parameters in (10) and (11), the order of convergence will reach at four and eight, respectively without using any more functional evaluations.

Theorem 1 *Let $f : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ has a simple zero ξ and is a sufficiently differentiable function in the closed interval $\mathbb{D}$ containing ξ. We also assume that initial guess $x = x_0$ is sufficiently close to ξ. Then, the iterative schemes* (10) *and* (11) *have fourth and eighth-order convergence, respectively when*

$$\beta_1 = 2(\alpha - 2), \quad \beta_2 = 2(\alpha - 1), \tag{12}$$

where $\alpha \in \mathbb{R}$.

Proof: Let us assume that the error at nth iteration is $e_n = x_n - \xi$. We expand $f(x_n)$ and $f'(x_n)$ around $x = \xi$ with the help of Taylor's series expansion. Then, we have

$$f(x_n) = f'(\xi)\left[\sum_{j=1}^{8} c_j e_n^j + O(e_n^9)\right], \tag{13}$$

and

$$f'(x_n) = f'(\xi)\left[\sum_{j=1}^{8} j c_j e_n^j + O(e_n^9)\right], \tag{14}$$

where $c_n = \dfrac{1}{n!}\dfrac{f^{(n)}(\xi)}{f'(\xi)}, \quad n = 2,\ 3,\ 4,\ \ldots, 8.$

By using the equations (13) and (14), we get

$$\begin{aligned} u =& e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_3c_2 - 4c_2^3 - 3c_4)e_n^4 + (8c_2^4 - 20c_3c_2^2 + 10c_4c_2 + 6c_3^2 \\ &- 4c_5)e_n^5 + \Big(52c_3c_2^3 - 16c_2^5 - 28c_4c_2^2 + (13c_5 - 33c_3^2)c_2 + 17c_3c_4 - 5c_6\Big)e_n^6 + O(e_n^7). \end{aligned} \tag{15}$$

By using the expression (15), we obtain

$$\begin{aligned} y_n - \xi =& c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_3c_2 + 3c_4)e_n^4 + (4c_5 - 8c_2^4 + 20c_3c_2^2 - 10c_4c_2 - 6c_3^2)e_n^5 \\ &+ (16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + (33c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6)e_n^6 - 2(16c_2^6 - 64c_3c_2^4 \\ &+ 36c_4c_2^3 + 9(7c_3^2 - 2c_5)c_2^2 + (8c_6 - 46c_3c_4)c_2 - 9c_3^3 + 6c_4^2 + 11c_3c_5 - 3c_7)e_n^7 + O(e_n^8). \end{aligned} \tag{16}$$

We have the following expansion of $f(y_n)$ about a point $x = \xi$

$$f(y_n) = f'(\xi)\Big[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 - 7c_3c_2 + 3c_4)e^4 - 2(6c_2^4 - 12c_3c_2^2 + 5c_4c_2 + 3c_3^2 - 2c_5)e_n^5 + \Big(16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + (33c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6\Big)e_n^6 - 2\Big(16c_2^6 - 64c_3c_2^4 + 36c_4c_2^3 + 9(7c_3^2 - 2c_5)c_2^2 - 9c_3^3 + (8c_6 - 46c_3c_4)c_2 + 6c_4^2 + 11c_3c_5 - 3c_7\Big)e_n^7 + O(e_n^8)\Big]. \tag{17}$$

With the help of expression (13) and (17), we get

$$v = c_2e_n + (2c_3 - 3c_2^2)e_n^2 + (8c_2^3 - 10c_3c_2 + 3c_4)e_n^3 + (37c_3c_2^2 - 20c_2^4 - 14c_4c_2 - 8c_3^2 + 4c_5)e_n^4 + (48c_2^5 - 118c_3c_2^3 + 51c_4c_2^2 + (55c_3^2 - 18c_5)c_2 - 22c_3c_4 + 5c_6)e_n^5 + (344c_3c_2^4 - 112c_2^6 - 163c_4c_2^3 + (65c_5 - 252c_3^2)c_2^2 + 2(75c_3c_4 - 11c_6)c_2 + 26c_3^3 - 15c_4^2 - 28c_3c_5 + 6c_7)e_n^6 + O(e_n^7). \tag{18}$$

Using equations (13) – (18), we have

$$L_f^*(x_n) = -(\beta_1 - \beta_2)c_2e_n + \frac{1}{2}\Big(2(\beta_1 - \beta_2)(\beta_2 + 3)c_2^2 + (-\beta_1 + \beta_2 - 2)c_2 + 4(\beta_2 - \beta_1)c_3\Big]e_n^2 + \sum_{r=0}^{5} P_r e_n^{r+3} + O(e_n^9), \tag{19}$$

where $P_r = P_r(\alpha, \beta_1, \beta_2, c_2, c_3, \ldots, c_8)$.
By inserting expressions (13) – (19) in the scheme (10), we obtain

$$e_{n+1} = \frac{1}{2}(2 + \beta_1 - \beta_2)c_2e_n^2 + \sum_{r=0}^{5} \bar{P}_r e_n^{r+3}, \tag{20}$$

where $\bar{P}_r = \bar{P}_r(\alpha, \beta_1, \beta_2, c_2, c_3, \ldots, c_8)$.
It is clear from the above expression (20) that we obtain at least third-order convergence, when we choose

$$\beta_2 = \beta_1 + 2. \tag{21}$$

Using the expression (21) in $P_0 = 0$, we obtain the following expression

$$(4 + \beta_1 - 2\alpha)c_2^2 = 0, \tag{22}$$

which further yields

$$\beta_1 = 2(\alpha - 2). \tag{23}$$

By substituting the expressions (21) and (23) in (10), we have

$$e_{n+1} = c_2[(\alpha - 1)c_2 + c_2^2 - c_3]e_n^4 + \sum_{r=2}^{5} \bar{P}_r e_n^{r+3} + O(e_n^9). \tag{24}$$

Again by using the expressions (13) – (19) and (21) – (23) in (11), we obtain

$$z_n - \xi = \bar{P}_1e_n^4 + \bar{P}_2e_n^5 + \bar{P}_3e_n^6 + \bar{P}_4e_n^7 + \bar{P}_5e_n^8 + O(e_n^9), \tag{25}$$

where $\bar{P}_1 = c_2[(\alpha-1)c_2+c_2^2-c_3]$, $\bar{P}_2 = \Big[2(3-4\alpha+\alpha^2)c_2^3+\frac{1}{2}c_2^2(\alpha+16c_3-1)+c_2\{4(\alpha-1)c_3-$

$2c_4\} - 4c_2^4 - 2c_3^2\Big]$, $\bar{P}_3 = \frac{1}{4}\Big[-8(7\alpha^2 - 19\alpha + 12)c_2^4 + 2c_2^3(2\alpha^2 - 9\alpha - 60c_3 + 7) + c_2^2\Big\{\alpha + 16(3\alpha^2 - 11\alpha + 8)c_3 + 48c_4 - 1\Big\} + 4c_2\Big\{2(\alpha-1)c_3 + 6(\alpha-1)c_4 + 18c_3^2 - 3c_5\Big\} + 4c_3(4(\alpha-1)c_3 - 7c_4) + 40c_2^5\Big]$, etc.

We can obtain the following Taylor series expansion from $f(z_n)$ about the point ξ with the help of expression (25)

$$f(z_n) = f'(\xi)\left[\bar{P}_1 e_n^4 + \bar{P}_2 e_n^5 + \bar{P}_3 e_n^6 + \bar{P}_4 e_n^7 + \left(\bar{P}_1^2 c_2 + \bar{P}_5\right) e_n^8 + O(e_n^9)\right]. \tag{26}$$

By using the equations (13) – (19) and (21) – (23) and (26), we have

$$\frac{u\Big(f(x_n) - f(z_n)\Big)f(y_n) + (x_n - z_n)(v-1)f(x_n)f(z_n)}{(2v-1)(x_n - z_n)f'(x_n)f(z_n) + \Big(uf[x_n, z_n] - f(z_n)\Big)f(y_n)} = e_n - c_2(c_2^3 - 2c_3c_2 + c_4)P_1 e_n^8 + O(e_n^9), \tag{27}$$

Finally by substituting the above expression in scheme (11), we obtain

$$e_{n+1} = c_2^2\left((\alpha-1)c_2 + c_2^2 - c_3\right)(c_2^3 - 2c_2c_3 + c_4)e_n^8 + O(e_n^9). \tag{28}$$

This above expressions (24) and (28) reveal that new constructions of Chebyshev-Halley methods (10) and (11) reach optimal fourth and eighth-order convergence, respectively. This completes the proof. □

Local Convergence

The local convergence of method (10) was given using hypotheses up to the fourth derivative of function f although only the first derivative appears in this method. The local convergence of method (11) requires the usage of the eighth derivative. These hypotheses limit the applicability of both methods. As a motivational example, define function f on $\mathbb{R}$, $\mathbb{D} = [-\frac{5}{2}, \frac{1}{2}]$ by

$$f(x) = \begin{cases} x^3 lnx^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Let us assume that our desired root is $\xi = 1$. Then, we have that

$$f'(x) = 3x^2 lnx^2 + 5x^4 - 4x^3 + 2x^2, \quad f'(1) = 3,$$

$$f''(x) = 12x lnx^2 + 20x^3 - 12x^2 + 10x$$

and

$$f'''(x) = 12lnx^2 + 60x^2 - 12x + 22.$$

Then, obviously, function $f'''(x)$ is unbounded on $\mathbb{D}$. Hence, the results in [21, 29], cannot apply to show the convergence of method (5) or its special cases requiring hypotheses on the third derivative of function F or higher. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations [1-32]. These results show that if the initial point x_0 is sufficiently close to the solution ξ, then the sequence $\{x_n\}$ converges to ξ. But how close to the solution ξ the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. The same technique can be used to other methods.

In this section, we present the local convergence analysis of method (5) using hypotheses only on the first derivative function f and Lipschitz constants. Similarly, we can the study method

(14). We expand the applicability of these methods in this way. Moreover, we provide computable radius of convergence, error estimates on the distances $|x_n - \xi|$ and a uniqueness result. Let $L_0 > 0$, $L > 0$, $M \geq 1$ and $\alpha, \beta_1\ \beta_2 \in \mathbb{R}$ be given constants. It is convenient for the local convergence analysis that follows to introduce some functions and parameters. Define functions $g_1,\ p$ and h_p on the interval $\left[0,\ \frac{1}{L_0}\right)$, by

$$g_1(t) = \frac{Lt}{2(1-L_0 t)},$$
$$p(t) = \begin{cases} \big(L_0 + 2|\beta_2 - 2\alpha|Mg_1(t) + 2M\big)t + \dfrac{\big(|\alpha\beta_2| + |\alpha+\beta_2|\big)M^2 g_1(t)}{1-\frac{L_0}{2}t}, & \text{if } 2|\beta_2 - 2\alpha| > |\beta_2| \\ \big(L_0 + 2(|\beta_2| + 2|\alpha|)|Mg_1(t) + 2M\big)t + \dfrac{\big(|\alpha\beta_2| + |\alpha+\beta_2|\big)M^2 g_1(t)}{1-\frac{L_0}{2}t}, & \text{if } 2|\beta_2 - 2\alpha| \leq |\beta_2| \end{cases}$$
$$h_p(t) = p(t) - 1,$$

and parameter r_1 by

$$r_1 = \frac{2}{2L_0 + L}.$$

Then, we have that $r_1 < \dfrac{1}{L_0}$ and for each $t \in [0,\ r_1),\ 0 \leq g_1(t) < 1$. Moreover, we have that $h_p(0) = -1$ and $h_p(t) \to +\infty$ as $t \to \dfrac{1^-}{L_0}$. It follows from the intermediate value theorem that function h_p has zeros in the interval $\left(0,\ \frac{1}{L_0}\right)$. Denote by r_p the smallest such zero. Furthermore, we also define the following functions g_2 and h_2 on the interval $[0,\ r_p)$, by

$$g_2(t) = \frac{1}{2(1-L_0 t)}\left[L + \frac{4M^3 g_1(t)\Big(|\beta_1 - \beta_2| + |\beta_2| g_1(t) t + t\Big)t}{(1-\frac{L_0}{2}t)(1-p(t))}\right]t,$$

and

$$h_2(t) = g_2(t) - 1.$$

We get that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_p^-$. Let r_2 be the smallest zero of function h_2 on the interval $(0,\ r_p)$.
Set

$$r = \min\{r_1,\ r_2\}. \tag{29}$$

Then, we have that

$$0 < r \leq r_1. \tag{30}$$

and for each $t \in [0,\ r)$,

$$0 \leq g_1(t) < 1, \tag{31}$$

$$0 \leq p(t) < 1, \tag{32}$$

and

$$0 \leq g_2(t) < 1. \tag{33}$$

Let $U(\gamma,\ \delta), \bar{U}(\gamma,\ \delta)$ denote, respectively for the open and closed balls in $\mathbb{R}$, with center $\gamma \in \mathbb{R}$, and of radius $\delta > 0$. Next, we present the local convergence analysis of method (4), (10) using the preceding notation.

Theorem 2 *Let $f : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that there exist $\xi \in \mathbb{D}$,*

$L_0 > 0$, $L > 0$, $M \geq 1$, $\alpha,\ \beta_1,\ \beta_2 \in \mathbb{R}$ such that for each $x,\ y \in \mathbb{D}$,

$$f(\xi) = 0, \quad f'(\xi)^{-1} \in L(\mathbb{R},\ \mathbb{R}), \tag{34}$$

$$|f'(\xi)^{-1}(f'(x) - f'(\xi)| \leq L_0|x - x_0|. \tag{35}$$

Let us assume that $\Omega_0 = \mathbb{D} \cap U\left(\xi, \frac{1}{L_0}\right)$.

$$|f'(\xi)^{-1}\Big(f'(x) - f'(y)\Big)| \leq L|x - y|, \quad foreach\ x, y \in \Omega_0, \tag{36}$$

$$|f'(\xi)^{-1}f'(x)| \leq M, \quad foreach\ x \in \Omega_0, \tag{37}$$

and

$$\bar{U}(\xi, r) \subset \mathbb{D}, \tag{38}$$

hold, where the convergence radius r is defined by (29). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(\xi,\ r) - \{\xi\}$ by method (5) is well defined, remains in $U(\xi,\ r)$ for each $n = 0, 1, 2, \ldots$ and converges to ξ. Moreover, the following estimates hold

$$|y_n - \xi| \leq g_1(|x_n - \xi|)|x_n - \xi| \leq |x_n - \xi| < r, \tag{39}$$

and

$$|z_n - \xi| \leq g_2(|x_n - \xi|)|x_n - \xi| < |x_n - \xi|, \tag{40}$$

where the "g" functions are defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$, the limit point ξ is the only solution of equation $f(x) = 0$ in $\Omega_1 = \bar{U}(\xi,\ T) \cap \mathbb{D}$.

Proof: We shall show estimates (39)–(40) using mathematical induction. By hypotheses $x_0 \in U(\xi, r) - \{\xi\}$, (29) and (35), we get

$$|f'(\xi)^{-1}(f'(x_0) - f'(\xi))| \leq L_0|x_0 - \xi| < L_0 r < 1. \tag{41}$$

It follows from the (41) and the Banach Lemma on invertible functions [3, 4, 26] that $f'(x_0) \neq 0$ and

$$|f'(x_0)^{-1}f'(\xi)| \leq \frac{1}{1 - L_0|x_0 - \xi|}. \tag{42}$$

Hence, y_0 is well defined by the first sub-step of the method (5) for $n = 0$. Then, we have by (29), (31), (36) and (42) that

$$\begin{aligned}
|y_0 - \xi| &= |x_0 - \xi - f'(x_0)^{-1}f(x_0)| \\
&\leq |f'(x_0)^{-1}f'(\xi)||\textstyle\int_0^1 f'(\xi)^{-1}\Big[f'(\xi + \theta(x_0 - \xi)) - f'(x_0)\Big](x_0 - \xi)d\theta| \\
&\leq \frac{L|x_0 - \xi|^2}{2(1 - L_0|x_0 - \xi|)} \\
&= g_1(|x_0 - \xi|)|x_0 - \xi| < |x_0 - \xi| < r,
\end{aligned} \tag{43}$$

which shows (39) for $n = 0$ and $y_0 \in U(\xi,\ r)$.

The fraction in (10) can be written for $n = 0$ as $\frac{N_0}{D_0}$, where $N_0 = 2f(y_0)[(\beta_1 - \beta_2)f'(x_0) + \beta f(y_0) + f(x_0)]$ and $D_0 = \frac{1}{2}f(x_0)D_1$, $D_1 = 2f(x_0) + 2(\beta_2 - 2\alpha)f(y_0) - \frac{(\alpha+\beta_2)f(y_0)f'(x_0)}{f(x_0)} - \alpha\beta_2\frac{[f(y_0)]^2}{f(x_0)} - f'(x_0)$.

We need to show that $f(x_0) \neq 0$, $f(x_0) - 2f'(x_0) \neq 0$, $f(x_0) + \beta_2 f(y_0) \neq 0$ and $D_1 \neq 0$ for $x_0 \neq \xi$. Using (29), (34) and (35), we have that

$$|\Big(f'(\xi)(x_0-\xi)\Big)^{-1}\Big(f(x_0) - f(\xi) - f'(\xi)(x_0-\xi)\Big)| \leq |x_0-\xi|^{-1}\frac{L_0}{2}|x-\xi|^2 = \frac{L_0}{2}|x_0-\xi| < 1. \tag{44}$$

It follows from (44) that $f(x_0) \neq 0$ and

$$|f'(x_0)^{-1}f'(\xi)| \leq \frac{1}{|x_0-\xi|\Big(1-\frac{L_0}{2}|x_0-\xi|\Big)}. \tag{45}$$

Then, by (29), (32), (35), (37), (43) and (45), we obtain for $2|\beta_2 - 2\alpha| > |\beta_2|$ and using the first version of the function p in turn that

$$\begin{aligned}
&\left|f'(\xi)^{-1}\left[f'(x_0) - f'(\xi) + \frac{\alpha\beta_2[f(y_0)]^2}{f(x_0)} + \frac{(\alpha+\beta_2)f(y_0)f'(x_0)}{f(x_0)} - 2f(x_0) - 2(\beta_2 - 2\alpha)f(y_0)\right]\right| \\
&\leq L_0|x_0-\xi| + \frac{|\alpha\beta_2|M^2|y_0-\xi|}{|x_0-\xi|\Big(1-\frac{L_0}{2}|x_0-\xi|\Big)} + \frac{|\alpha+\beta_2|M^2|y_0-\xi|}{|x_0-\xi|\Big(1-\frac{L_0}{2}|x_0-\xi|\Big)} \\
&\quad + 2M|x_0-\xi| + 2|\beta_2-2\alpha|M|y_0-\xi| \\
&\leq L_0|x_0-\xi| + \frac{|\alpha\beta_2|M^2 g_1\Big(|x_0-\xi|\Big)}{1-\frac{L_0}{2}|x_0-\xi|} + \frac{|\alpha+\beta_2|M^2 g_1\Big(|x_0-\xi|\Big)}{1-\frac{L_0}{2}|x_0-\xi|} \\
&\quad + 2\Big[M + |\beta_2-2\alpha|M g_1\Big(|x_0-\xi|\Big)\Big]|x_0-\xi| \\
&= p\Big(|x_0-\xi|\Big) < p(r) < 1.
\end{aligned} \tag{46}$$

Hence, we get that

$$|D_0^{-1}f'(\xi)| \leq \frac{1}{1-p\Big(|x_0-\xi|\Big)}. \tag{47}$$

If we use $2|\beta_2 - 2\alpha| \leq |\beta_2|$ then the term $2|\beta_2 - 2\alpha|$ can be replaced by $2(|\beta_2| + 2|\alpha|)$. For this condition, we use the second version of the function p. Further, by using (45) and (47) we have that

$$|D_1^{-1}f'(\xi)| \leq \frac{2}{|x_0-\xi|\Big(1-\frac{L_0}{2}|x_0-\xi|\Big)\Big(1-p\Big(|x_0-\xi|\Big)\Big)}. \tag{48}$$

Hence, x_1 is well defined by (10) for $n = 0$. We also notice from equation (46) which implies that $f(x_0) - 2f'(x_0) \neq 0$, $f(x_0) + \beta_2 f(y_0) \neq 0$. Further, we have

$$|f'(\xi)^{-1}\Big(f'(x_0) - f'(\xi)\Big) - \frac{f(x_0)}{2}| \leq L_0|x_0-\xi| + \frac{M|x_0-\xi|}{2} \leq p(|x_0-\xi|) < p(r) < 1, \tag{49}$$

and

$$\begin{aligned}
&|\Big(f'(\xi)(x_0-\xi)\Big)^{-1}\Big((f(x_0) - f(\xi) - f'(\xi)(x_0-\xi)) + \beta_2 f(y_0)\Big)| \\
&\leq |x_0-\xi|^{-1}\left(\frac{L_0}{2}|x_0-\xi|^2 + |\beta_2|M|y_0-\xi|\right) \\
&\leq \frac{L_0}{2}|x_0-\xi| + M|\beta_2|g_1|\Big(|x_0-\xi|\Big) \\
&\leq p\Big(|x_0-\xi|\Big) < p(r) < 1,
\end{aligned} \tag{50}$$

for p given by the first formula, if $2|\beta_2-2\alpha| > |\beta_2|$ and from the second formula if $2|\beta_2-2\alpha| \le |\beta_2|$.
Now, we need to estimate the following

$$\begin{aligned}|N_0| &\le 2|f'(\xi)^{-1}f(y_0)|\Big[|\beta_1-\beta_2|+|\beta_2||f'(\xi)^{-1}f(y_0)|+|f'(\xi)^{-1}f(x_0)|\Big]\\ &\le 2M|y_0-\xi|\Big[|\beta_1-\beta_2|+|\beta_2|M|y_0-\xi|+M|x_0-\xi|\Big]\\ &\le 2Mg_1\Big(|x_0-\xi|\Big)|x_0-\xi|\Big[|\beta_1-\beta_2|+|\beta_2|Mg_1\Big(|x_0-\xi|\Big)|x_0-\xi|+M|x_0-\xi|\Big].\end{aligned} \tag{51}$$

Therefore, by (10) (for $n=0$), (30), (33), (37), (42), (43), (45), (47), (48), and (51) we get in turn that

$$\begin{aligned}|x_1-\xi| &\le |x_0-\xi-f'(x_0)^{-1}f(x_0)|+\frac{2M^3|y_0-\xi|(|\beta_1-\beta_2|+|\beta_2||y_0-\xi|+|x_0-\xi|)|x_0-\xi|}{(1-L_0|x_0-\xi|)(1-\frac{L_0}{2}|x_0-\xi|)(1-p(|x_0-\xi|))|x_0-\xi|}\\ &\le \frac{2M^3g_1(|x_0-\xi|)\left[|\beta_1-\beta_2|+|\beta_2|g_1(|x_0-\xi|)|x_0-\xi|+|x_0-\xi|\right]|x_0-\xi|^3}{(1-L_0|x_0-\xi|)(1-\frac{L_0}{2}|x_0-\xi|)(1-p(|x_0-\xi|))}\\ &\quad+\frac{L|x_0-\xi|^2}{2(1-L_0|x_0-\xi|)}\\ &= g_2(|x_0-\xi|)|x_0-\xi| < |x_0-\xi| < r,\end{aligned} \tag{52}$$

which shows (40) for $n=0$ and $x_1 \in U(\xi\ r)$. By simply replacing x_0, y_0 by x_k, y_k in the preceding estimates we arrive at (39)–(40). Using the estimates $\|x_{k+1}-\xi\| < \|x_k-\xi\| < r$, we deduce that $\lim_{k\to\infty} x_k = \xi$ and $x_{k+1} \in \Omega_1$. Finally, to show the uniqueness part, let $Q = \int_0^1 f'(y^*+\theta(\xi-y^*))d\theta$ for some $y^* \in \Omega_1$ with $f(y^*)=0$. Using (35), we get that

$$\begin{aligned}&\|f'(\xi)^{-1}(Q-f'(\xi))\| \le \|\textstyle\int_0^1 L_0|y^*+\theta(\xi-y^*)-\xi\|d\theta\\ &\le \textstyle\int_0^1(1-t)\|y^*-\xi\|d\theta \le \dfrac{L_0}{2}T < 1.\end{aligned} \tag{53}$$

It follows from (53) that Q is invertible. Then, in view of the identity $0 = f(\xi)-f(y^*) = Q(\xi-y^*)$, we conclude that $\xi = y^*$. □

Remarks

(a) In view of (35) and the estimate

$$\begin{aligned}|f'(\xi)^{-1}f'(x)| &= |f'(\xi)^{-1}(f'(x)-f'(\xi))+I|\\ &\le 1+|f'(\xi)^{-1}(f'(x)-f'(\xi))|\\ &\le 1+L_0|x_0-\xi|\end{aligned}$$

condition (37) can be dropped and M can be replaced by

$$M(t) = 1+L_0t$$

or by $M(t)=M=2$, since $t\in[0,\frac{1}{L_0})$.

(b) The results obtained here can be used for operators f satisfying the autonomous differential equation [3, 4] of the form

$$f'(x) = P(f(x)),$$

where P is a known continuous operator. Since $f'(\xi) = P(f(\xi)) = P(0)$, we can apply the results without actually knowing the solution ξ. Let as an example $f(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$.

(c) The radius r_1 was shown in [3, 4] to be the convergence radius for Newton's method under conditions (35) and (36). It follows from (31) and the definition of r_1 that the convergence radius r of the method (5) cannot be larger than the convergence radius r_1 of the second order Newton's method. As already noted in r_1 is at least as the convergence ball give by Rheinboldt [28]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_1$$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L} \to 0.$$

That is our convergence ball r_1 is at most three times larger than Rheinboldt's. The same value for r_R given by Traub [30].

Numerical experiments

In this section, we want to assert that our proposed an optimal family of Chebyshev Halley methods perform better than the families of Chebyshev-Halley methods proposed by Li et al. (2014) and Sharma (2015). Some of the researchers who want to assert that their methods are superior than the other existing methods available in the literature. Generally, they consider either some well-known or standard or self made test problem and then mold the initial approximation to assert that their methods are superior than other methods. Molding the initial guess mean, let A researcher who wants to compare his/her method/methods with B's method/methods by considering a particular test problem with x_0 as initial guess. Now, if A's method/methods does/do not perform better than B's method/methods then A changes the initial guess and continue this process until he/she gets better results than B's method/methods. If A does not get success on that particular test problem on any initial guess then A consider another test problem and continue the same process until A does not get success.

To halt this practice, we consider total six numerical examples out of them first two are chosen from Li et al. [21], third and fourth from Sharma [29] with same initial guesses which are mentioned in their papers. Further, fifth and six test examples are taken from Petkovíc et al. [25]. The details of chosen test problems or functions are available in the Table 1. Further, the initial approximations and zeros of the corresponding test functions are also display in the same table.

To check the effectiveness and validity of the theoretical results, we employ the new optimal family of Chebyshev–Halley methods (11) $(MCHM)$, with Chebyshev's method (MCM) $(\alpha = 0)$, Halley's method (MHM) $(\alpha = \frac{1}{2})$ and super-Halley method $(MSHM)$ $(\alpha = 1)$. We shall compare our schemes with a family of Chebyshev-Halley type methods that is very recently proposed by Li et al. [21], out of them we shall pick their best methods (which are claimed by them not by us) namely, Chebyshev's method (LCM) $(\alpha = 0)$, Halley's method (LHM) $(\alpha = \frac{1}{2})$ and super-Halley method $(LSHM)$ $(\alpha = 1)$. Finally, we shall also compare our schemes with the improved Chebyshev-Halley methods which is developed by Sharma [29], between them we shall choose their best methods namely, ($\alpha = 0,\ \alpha = \frac{1}{2},\ \alpha = 1$) denoted by $SCM,\ SHM$, and $SSHM$, respectively.

In the Table 2, we display the minimum number of iterations are required to get the desire accuracy to the corresponding zeros of the functions $f_1(x) - f_6(x)$ which are given in Table 1. In addition, we also exhibit the absolute errors $|x_{n+1} - x_n|$ for first three consecutive approximations in this table. Further, the meaning of $(Ae - h)$ is $(A \times 10^{-h})$. Furthermore, we also want to demonstrate the theoretical order of convergence which is proved in section 3. Therefore, to calculate the computational order of convergence, we use the following formula proposed by [31], which is defined as follows

$$\rho \approx \frac{\ln |(x_{n+1} - \xi)/(x_n - \xi)|}{\ln |(x_n - \xi)/(x_{n-1} - \xi)|}.$$

But, this COC requires the exact root ξ and there are many practical situations where the exact root is not known in advance. To overcome this problem, Grau-Sánchez et al. [13], given another definition of COC, which is defined as follows

$$\rho \approx \frac{\ln |\check{e}_{n+1}/\check{e}_n|}{\ln |\check{e}_n/\check{e}_{n-1}|}, \tag{54}$$

where $\check{e}_n = x_n - x_{n-1}$.
All computations have been performed by using the programming package $Mathematica$ 9 with multiple precision arithmetic. We use $\epsilon = 10^{-300}$ as a tolerance error. The following stopping criteria are chosen for computer programs:
$(i)|x_{n+1} - x_n| < \epsilon$ and $(ii)|f(x_{n+1})| < \epsilon$.
It is noteworthy from the table 2, that our proposed schemes perform better than the Li et al. (2014) and Sharma (2015), when the accuracy is tested in the high precision digits. For better comparison, we give a column by column comparison of different modifications of Chebyshev–Halley methods, so that we can easily see the exact difference between the proposed modifications and existing modifications of Chebyshev–Halley methods. Further, the accuracy in numerical values of approximations to the root by the proposed scheme is higher than the recently improvement of Chebyshev–Halley methods given by Li et al. [21] and Sharma [29]. In general, the our optimal family of Chebyshev-Halley methods $(MCHM)$ is superior among all the other proposed methods. This superiority is in accordance because it is an optimal modification of Chebyshev-Halley methods according to Kung-Traub conjecture [20], which is discussed in the previous section. The computational order of convergence (COC) and dynamic study of these methods also confirmed the above conclusions to a great extent.

Now, we also demonstrate the theoretical results which we proposed in section 4, by the applying on some other numerical examples, which are defined as follows:

Example 1 *Let f be a function defined on $\mathbb{D} = \bar{U}(0,\ 1)$, which is given as follows*

$$f_7(x) = e^x - 1. \tag{55}$$

Then the first-derivative is $f_7'(x) = e^x$. We get that $L_0 = e - 1 < L = e^{\frac{1}{e-1}}$, $\alpha = 1$ $M = 2$, $\beta_1 = 2(\alpha - 2)$ and $\beta_2 = 2(\alpha - 1)$. By substituting different values of parameters, we get different radius of convergence which are display in the Table 3.

Table 1: Test problems

$f(x)$	Initial guess	$Root(\xi)$
$f_1(x) = 10xe^{-x^2} - 1$; (see [21])	1.7	$1.67963061042845\ldots$
$f_2(x) = (x+2)e^x - 1$; (see [21])	-0.5	$-0.442854401002389\ldots$
$f_3(x) = e^x + 2^{-x} + 2\cos x - 6$; (see [29])	3.5	$1.82938360193385\ldots$
$f_4(x) = (x-2)(x^{10} + x + 1)e^{-x-x}$; (see [29])	2.5	2
$f_5(x) = e^{x^2+7x-30} - 1$; (see [25])	3.3	3
$f_6(x) = (x-2)^2 - \log x - 33x$; (see [25])	33	$36.9894735829447\ldots$

Example 2 *Returning back to the motivation example at the introduction on this section, we have $L = L_0 = 146.6629073$, $M = 2$, $\beta_1 = 2(\alpha - 2)$ and $\beta_2 = 2(\alpha - 1)$. By substituting different values of parameters, we get different radius of convergence which are display in the Table 4.*

Example 3 Continuous stirred tank reactor (CSTR)

Let us consider the isothermal continuous stirred tank reactor (CSTR). Components A and R are fed to the reactor at rates of Q and $q - Q$, respectively. Then, we obtain the following reaction scheme in the reactor (for the details see [10]):

$$\begin{aligned} A + R &\to B \\ B + R &\to C \\ C + R &\to D \\ C + R &\to E \end{aligned}$$

The problem was analyzed by Douglas [12] in order to design simple feedback control systems. He presented the following expression for the transfer function of the reactor

$$K_C \frac{2.98(x + 2.25)}{(x + 1.45)(x + 2.85)^2(x + 4.35)} = -1,$$

where K_C is the gain of the proportional controller. The control system is stable for values of K_C that yields roots of the transfer function having negative real part. If we choose $K_C = 0$ we get the poles of the open-loop transfer function as roots of the nonlinear equation:

$$f_8(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875. \tag{56}$$

No doubts, the above function f_1 has four zeros $x^* = -1.45, -2.85, -2.85, -4.35$. However, our required zero is $x^* = -4.35$ for expression (57). Let us also consider $\mathbb{D} = [-4.5, -4]$.

Then, we obtain

$$L_0 = L = 2.760568793, \ M = 2$$

Now, with the help of different values, we get different radius of convergence displayed in Table 5.

Table 2: (Comparison of different multi-point methods)

$f(x)$		LCM	LHM	$LSHM$	SCM	SHM	$SSHM$	MCM	MHM	$MSHM$
	$\|x_2-x_1\|$	1.4e−1	7.6e−1	5.9e−1	2.5e−1	1.9e−1	8.1e−1	6.3e−3	3.8e−3	3.2e−2
$f_1(x)$	$\|x_3-x_2\|$	4.2e−5	4.5e−2	4.2e−3	1.3e−4	1.3e−5	1.4e−10	8.1e−20	1.2e−22	2.9e−14
	$\|x_4-x_3\|$	5.7e−23	4.2e−8	4.2e−6	8.8e−24	2.3e−30	1.4e−80	5.7e−155	1.6e−178	1.1e−110
	n	6	7	6	6	6	5	5	5	5
	ρ	5.000	5.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
	$\|x_2-x_1\|$	1.5e−2	1.1e+0	2.1e−1	1.1e−1	9.0e−2	4.8e−2	2.3e−2	2.3e−2	2.6e−2
$f_2(x)$	$\|x_3-x_2\|$	2.5e−7	2.0e−1	1.4e−2	8.0e−4	1.7e−5	9.0e−8	9.3e−11	1.2e−10	2.9e−10
	$\|x_4-x_3\|$	2.0e−31	1.1e+0	1.6e−9	1.1e−15	2.2e−20	2.7e−53	7.0e−78	7.3e−77	1.1e−73
	n	96	6	6	6	6	5	5	5	5
	ρ	5.000	5.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
	$\|x_2-x_1\|$	8.9e−10	3.7e−14	5.4e−12	2.7e−10	6.4e−11	6.2e−15	8.8e−15	6.3e−15	3.9e−15
$f_3(x)$	$\|x_3-x_2\|$	7.3e−47	4.3e−49	1.4e−69	1.3e−57	5.6e−62	4.0e−115	9.2e−114	4.9e−115	7.2e−117
	$\|x_4-x_3\|$	2.7e−232	9.0e−244	3.9e−415	1.3e−341	2.5e−368	1.1e−916	1.4e−905	1.4e−916	8.8e−931
	n	5	5	4	4	4	4	4	4	4
	ρ	5.000	5.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
	$\|x_2-x_1\|$	2.0e−7	8.7e−8	1.2e−9	2.7e−8	6.1e−9	2.4e−12	4.4e−13	1.4e−13	1.5e−13
$f_4(x)$	$\|x_3-x_2\|$	8.3e−35	7.2e−37	1.4e−55	2.7e−46	8.8e−51	2.3e−95	5.1e−102	1.3e−106	2.8e−106
	$\|x_4-x_3\|$	1.1e−171	2.7e−182	2.8e−331	2.5e−274	7.4e−302	1.6e−759	1.6e−813	9.0e−851	4.4e−848
	n	5	5	4	5	4	4	4	4	4
	ρ	5.000	5.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
	$\|x_2-x_1\|$	D	1.3e−2	1.1e+0	1.2e−1	1.2e−1	1.0e+0	3.6e−2	3.6e−2	3.7e−2
$f_5(x)$	$\|x_3-x_2\|$	D	9.2e−3	2.9e−1	1.6e−2	8.7e−3	5.7e−4	1.9e−9	2.1e−9	3.1e−9
	$\|x_4-x_3\|$	D	5.9e−3	2.3e−6	4.8e−7	4.8e−9	1.4e−21	5.6e−68	1.6e−67	5.2e−66
	n	D	D	7	7	7	6	5	5	5
	ρ	NC	5.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000
	$\|x_2-x_1\|$	7.2e−4	1.2e−4	1.7e−6	7.2e−4	1.2e−4	1.7e−6	2.4e−6	9.3e−6	1.7e−6
$f_6(x)$	$\|x_3-x_2\|$	5.7e−26	2.8e−30	9.4e−47	7.8e−27	3.6e−32	7.7e−58	3.8e−47	1.1e−50	7.7e−58
	$\|x_4-x_3\|$	1.5e−136	2.3e−158	2.4e−288	1.3e−164	3.3e−197	1.4e−468	1.8e−381	3.1e−410	1.4e−468
	n	5	5	5	5	5	4	4	4	4
	ρ	5.000	5.000	6.000	6.000	6.000	8.000	8.000	8.000	8.000

(D: stands for divergence. NC means no need to calculate.)

Table 3: Behavior of scheme (10) **on example** (1)

α	r_1	r_2	$r = \min\{r_1,\ r_2\}$	x_0	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	ρ
$\alpha = 0$	0.382692	0.066575	0.066575	0.65	3.4e(−6)	2.8e(−23)	1.3e(−91)	4.000
$\alpha = \frac{1}{2}$	0.382692	0.0877379	0.0877379	0.85	4.1e(−6)	2.3e(−23)	2.1e(−92)	4.000
$\alpha = 1$	0.382692	0.0877468	0.0877468	0.86	2.2e(−6)	1.0e(−24)	4.2e(−98)	4.000

Table 4: Behavior of scheme (10) **on example** (2)

α	r_1	r_2	$r = \min\{r_1,\ r_2\}$	x_0	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	ρ
$\alpha = 0$	0.00454557	0.00109685	0.00109685	1.0009	5.5e(−12)	7.4e(−45)	2.5e(−176)	4.000
$\alpha = \frac{1}{2}$	0.00454557	0.00167861	0.00167861	1.0011	1.9e(−11)	1.6e(−42)	7.6e(−167)	4.000
$\alpha = 1$	0.00454557	0.00167861	0.00167861	1.0011	2.5e(−11)	7.0e(−42)	4.1e(−164)	4.000

Example 4 In the study of the multi-factor effect, the trajectory of an electron in the air gap between two parallel plates is given by

$$x(t) = x_0 + \left(v_0 + e\frac{E_0}{m\omega}\sin(\omega t_0 + \alpha)\right)(t - t_0) + e\frac{E_0}{m\omega^2}\Big(\cos(\omega t + \alpha) + \sin(\omega + \alpha)\Big), \tag{57}$$

where e and m are the charge and the mass of the electron at rest, x_0 and v_0 are the position and velocity of the electron at time t_0 and $E_0 \sin(\omega t + \alpha)$ is the RF electric field between the plates. We choose the particulars parameters in the expression (57) in order to deal with a simpler expression, which is defined as follows:

$$f_9(x) = x - \frac{1}{2}\cos(x) + \frac{\pi}{4}. \tag{58}$$

The required zero of the above function $\alpha = -0.3090932715417949527419868089 24$.

Then, we have

$$L_0 = L = M = 1.523542095.$$

So, we obtain the different radius of convergence which are displayed in Table 6 by using the above values.

Attractor basins in the complex plane

In this section, we present the dynamics of the proposed method based on visual display of their basins of attraction when $f(x)$ is a given fixed complex polynomial $q(z)$. We further

Table 5: Behavior of scheme (10) **on example** (3)

α	r_1	r_2	$r = \min\{r_1,\ r_2\}$	x_0	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	ρ
$\alpha = 0$	0.241496	0.0473699	0.0473699	−4.396	2.1e(−5)	1.1e(−18)	8.4e(−72)	4.000
$\alpha = \frac{1}{2}$	0.241496	0.0650116	0.0650116	−4.41	4.5e(−5)	1.9e(−17)	5.9e(−67)	4.000
$\alpha = 1$	0.241496	0.0650156	0.0650156	−4.41	3.2e(−5)	3.6e(−18)	5.3e(−70)	4.000

Table 6: Behavior of scheme (10) **on example** (4)

α	r_1	r_2	$r = \min\{r_1,\ r_2\}$	x_0	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	ρ
$\alpha = 0$	0.437577	0.066575	0.066575	-0.374	1.2e(-6)	1.5e(-25)	3.1e(-101)	4.000
$\alpha = \frac{1}{2}$	0.437577	0.124466	0.124466	-0.419	4.1e(-6)	7.5e(-24)	8.3e(-95)	4.000
$\alpha = 1$	0.382692	0.124493	0.124493	-0.419	1.9e(-6)	1.8e(-25)	1.4e(-101)	4.000

investigate some dynamical properties of the attained simple root finders in the complex plane by analyzing the structure of their basins of attraction. It is known that the corresponding fractal of an iterative root-finding method is a boundary set in the complex plane, which is characterized by the iterative method applied to a fixed polynomial $q(z) \in \mathbb{C}$, see e.g. [27, 24, 7, 6].

The aim herein is to use basin of attraction as another way for comparing the iterative methods. Therefore, we here investigate the dynamics of the listed simple root finders in the complex plane using basins of attraction which gives important information about convergence and stability of the method. To start with, let us recall some basic concepts which are related to basins of attractions. To start with, let us recall some basic concepts which are related to basins of attractions. Let $g : \mathbb{C} \to \mathbb{C}$ be a rational map on the Riemann sphere. The orbit of a point $z \in \mathbb{C}$ under g is defined

$$\{z,\ g(z),\ g^2(z),\ \ldots,\ g^n(z),\ \ldots\},$$

which consists of successive images of z by the rational map g. The dynamic behavior of the orbit of a point of g would be characterize by its asymptotic behavior. We first introduce some notions of a point in the orbit under g: a point $z_0 \in \mathbb{C}$ is known as a *fixed point* of g, if $g(z_0) = z_0$. In addition, z_0 is known as a *periodic point of period* $m > 1$, if $g^m(z_0) = z_0$, where m is smallest such integer. Further, if z_0 is a periodic point of period m then it is a fixed point for g^m. Moreover, there are mainly four types of fixed points of a map g, which are based on the magnitude of the derivative. A fixed point z_0 is known as:

If ξ is a root of $f(x)$, then the basin of attraction of ξ, is the collection of those initial approximations x_0 which converge to ξ. It is mathematically defined as follows:

$$B(\xi) = \{z_0 \in \mathbb{C} :\ g^n(z_0) \to \xi \text{ as } n \to \infty\}.$$

Arthur Cayley was the first person who considered the concept of the basins of attraction for Newton's method in 1879. Initially, he considered this concept for the quadratic polynomial. After some time, he also considered cubic polynomials, but was unable to find an obvious division for the basins of attraction as he earlier defined for the quadratic equations. In the early of 20th century, the French mathematicians Gaston Julia and Pierre Fatou started to understand the nature of complex cubic polynomials. The Julia set of a nonlinear map $g(z)$, called $J(g)$, is the closure of the set of its repelling fixed points and establishes the borders between the basins of attraction. On the other hand, the complement of $J(g)$ is known as the Fatou set $F(g)$. In simple words, the basins of attraction of any fixed point belongs to the Fatou set $F(g)$ and the boundaries of these basins of attraction belong to the Julia set $J(g)$. For the details of these concepts please see [11, 27, 24]. The aim herein is to use the basins of attraction as another way for characterizing initial approximations converging to the desired root ξ for the listed iteration algorithms. That is to say, the basins of attraction play a role representing a valuable dynamics of the iteration schemes under consideration.

In order to achieve a vivid description from a dynamical point of view, we consider a rectangle $\mathbb{D} = [-3,\ 3] \times [-3,\ 3] \in \mathbb{C}$ with a 400×400 grid, and we assign a color to each point $z_0 \in D$ according to the simple root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge. In this section, we consider the stopping criterion for convergence to be less than 10^{-4} wherein the maximum number of full cycles for each method is considered to be 200. In this way, we distinguish the attraction basins by their colors for different methods. For concrete examples of dynamics of the listed methods behind the basins of attraction, we present several test problems described below.
Test problem 1. Let $p_1(z) = (z^4+1)$, having simple zeros $\{-0.707107-0.707107i,\ -0.707107+0.707107i,\ 0.707107-0.707107i,\ 0.707107+0.707107i\}$. It is straight forward to see from Fig. 1 – 3 that our methods, namely MCM, MHM and $MSHM$ are the best methods in terms of less chaotic behavior to obtain the solutions. Further, our methods also have the largest basins for the solution and is faster in comparison to all the mentioned methods.
Test problem 2. Let $p_2(z) = (z^3+2z)$, having simple zeros $\{0,\ -1.41421i,\ 1.41421i\}$. Based on Fig. 4 – 6 , it is observe that our proposed methods namely, MCM, MHM and $MSHM$ are the best methods because they have larger and brighter basin of attraction in comparison to the methods namely, LCM, LHM, $LSHM$, SCM, SHM and $SSHM$, respectively.

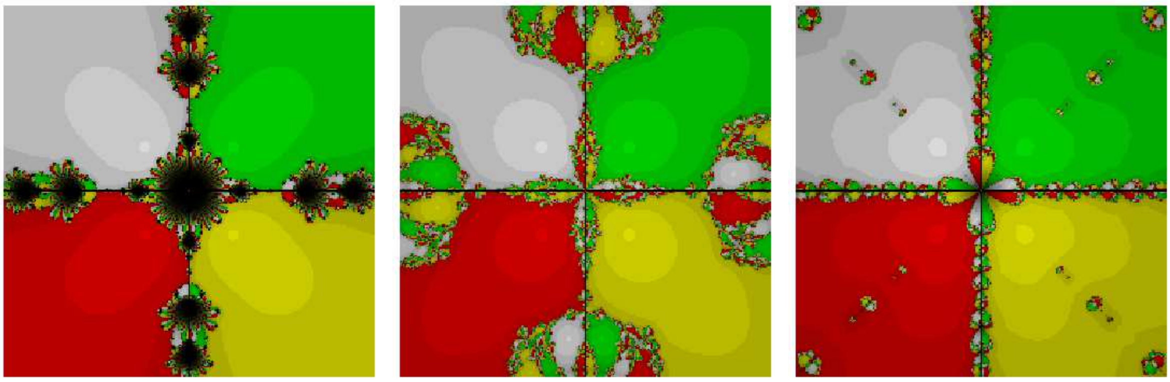

Figure 1: The methods LCM, LHM and $LSHM$, respectively for test problem 1.

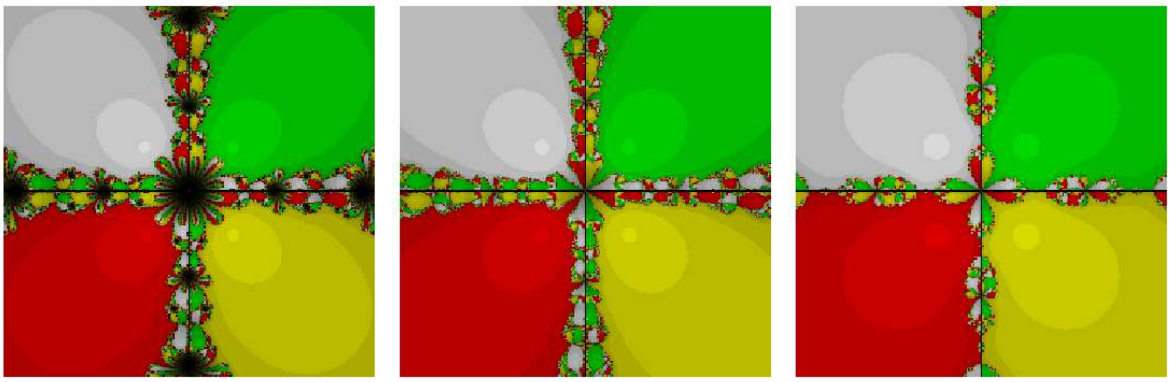

Figure 2: The methods SCM, SHM and $SSHM$, respectively for test problem 1.

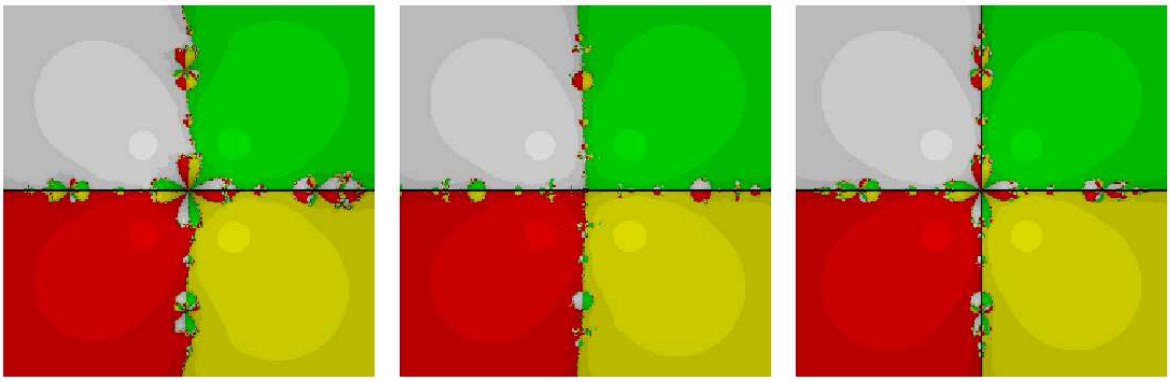

Figure 3: The methods MCM, MHM and $MSHM$, respectively for test problem 1.

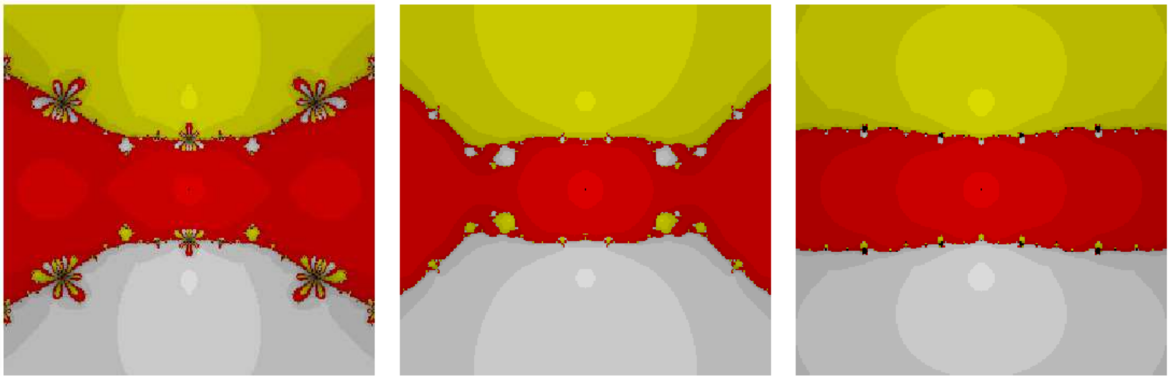

Figure 4: The methods LCM, LHM and $LSHM$, respectively for test problem 2.

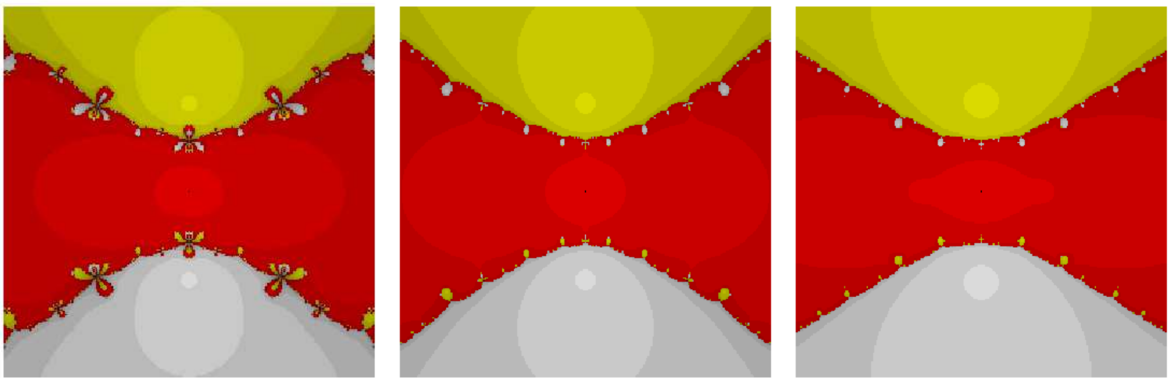

Figure 5: The methods SCM, SHM and $SSHM$, respectively for test problem 2.

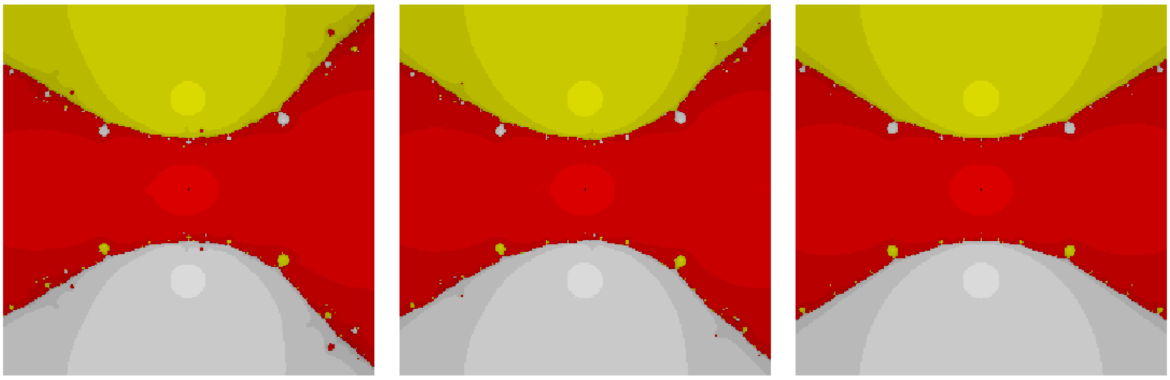

Figure 6: The methods MCM, MHM and $MSHM$, respectively for test problem 2.

Conclusions

The present contribution of this study is not only to increase the order of convergence of classical Chebyshev-Halley method. But, we also provide theoretical radius of convergence which guaranteed for the convergence of iterative methods. In addition, our schemes (10) and (11) can further produce many more new optimal methods of order four and eight, respectively for each α. On the other hand, Sharma [29] and Li at al. [21] didn't get guess in order to obtain optimal methods for each α in their studies. On the accounts of results obtained in the Table 2, it can be concluded that the proposed methods are highly efficient as compared to the existing methods in term of computational efficiency and speed. We are claiming the superiority of our methods because we compare them on the same test problems with same initial approximations which are they taken in their papers (for detail please see Table 1). We also verify in section 4 that these methods converge to the required root even though the third derivative is not bounded. Finally, the dynamical behaviors of our methods also demonstrate the superiority to the other known methods in terms of larger and brighter basin of attraction and less chaotic.

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