Winkler model for seismic responses of shafts under stochastic earthquakes

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Abstract

A simplified method for calculating the seismic responses of the shaft is proposed in this paper. First, based on the theory of Winkler elastic foundation beam, the urban shaft is simplified as a vertical beam. Secondly, the horizontal soil reaction and vertical shear tractions between the shaft circumference and the surrounding soils are considered through horizontal springs and rotating springs on the sidewall of the shaft. The translation and rocking motion of the shaft are considered through horizontal springs and rotating springs at the bottom of the shaft. Then, the dynamic analysis model of the shafts under seismic motion is established, and the control equation of the dynamic response of the shaft in frequency domain is deduced. Finally, the analytical solution of the steady state response of the shaft is obtained. Considering the randomness of the earthquake motion, this method can get the shaft responses under different ground motions efficiently. At the same time, the influence of ground motion frequency on the dynamic response of shaft can be observed.

Keywords: **Shaft, Winkler model, Seismic responses, Random**

1. Introduction

As a subsidiary structure connecting the ground and underground structures, the vertical shaft has been widely constructed in the areas of underground transportation system, power system and utility tunnel system. At present, round and square are mainly cross section shape of existing shafts, which are usually of small cross-section and shallow depth within 40 meters. With the exploitation and utilization of deep underground space in urban cities, large-depth (more than 40 meters) shafts are now widely used in deep urban drainage systems. Such as the Metropolitan Area Outer Underground Discharge Channel in Tokyo and the Deep Storage and Drainage Pipe System in Shanghai. Round is the mainly cross section shape of the deep shaft. The dynamic responses of the shafts are being studied.

In order to know the shaft dynamic responses, there are about two methods to calculate the dynamic responses: the quasi-static method and the three dimensional dynamic time history analysis method. In the quasi-static methods the shafts are usually treated as a vertical beam embedded in the soil [1]-[3]. In the dynamic time history analysis the results are more accurate. The main differences between the two methods is that the quasi-static method could not considering the shaft responses at every moment under the excitation of the earthquake motion, while the numerical methods such as three dimensional finite element dynamic time history analysis method and so on are usually with low computational efficiency and could not explain the dynamic responses from the perspective of mechanical mechanism for engineering design. Due to the uncertainty and complexity of the earthquake motion, the ground motion is a complicated time process which should be carefully considered when implementing the shaft seismic design. Therefore it is essential to propose a simplified method which can not only capture the dynamic responses of the shafts but also can calculate efficiently.

In this paper, a dynamic winkle beam physical model for the shaft is purposed and established with considering the horizontal reaction and vertical shear tractions from the surrounding soil and horizontal traction springs and rotating springs at the bottom of the shaft for the kinematic motion based on the existing quasit-static method. The dynamic equilibrium equation of the shaft is derived and the analytic solution of the kinematic equilibrium is solved and presented in frequency domain. The dynamic responses of the shafts under stochastic earthquakes excitation in time domain would be obtained efficiently through the FFT and iFFT with this simplified method.

2. Physical model

In the physical model of quasi-static method [3], horizontal reactions and vertical shear tractions distributed along the shaft wall are represent by the horizontal spring and the rotation spring. In order to reflect the translation motion and rotational motion, the shear spring and rotation spring are stetted at the shaft bottom.

Gerolymos and Gazetas [4]-[6] purposed a winkle four spring model for lateral response of rigid caisson foundations in liner soil. Due to the structural and functional characteristics of the caisson foundation, the caissons are usually simplified as a rigid body due to its great structural stiffness relative to the surrounding soil, while the pile foundations are usually simplified as a beam due to the small structural stiffness relative to the surrounding soil. There are many differences between the caisson foundation and the deep shaft, especially the structural stiffness, the underground deep shafts are hollow structures while the caisson foundations are solid one. Chen and Zhang [7] and Mayoral [8] concluded that the large-depth shaft dynamic responses in soft soil approximately like a rigid body with translation motion, rotational motion and small bending deformation. Considering the structural and functional characteristic of the shaft and effect of the soil-structure relative stiffness on the dynamic responses of the underground structure. Finally, the shaft is simplified as a beam. In this paper the simplified dynamic analysis method for the shaft is proposed with four springs and dashpots and the shaft is simplified a beam, as shown in Fig. 1

A circle in plain shaft embedded in homogeneous isotropic and viscoelastic soil was illustrated in Fig. 1. The assumption of model are as follows: the shaft is assumed to be with uniform wall thickness and the shaft with linear deformation under the seismic motion; the shaft is perfectly glued to the surrounding soils, indicating that there is no slippage or separation along the shaft-soil interface.

The parameters of the shaft are as follows: the shaft depth is L, the external diameter is D, the inner diameter is d, the Young's modulus is E, and the density is ρ. The parameters of the soil are as follows: the Young's modulus is E_s , the shear modulus is G_s , the density is ρ_s, the shear wave velocity is V_s. The four-spring coefficients purposed by Gazetas [9] is adopted for simulating the soil-caisson interaction here and will be introduced in the next session. The distributed lateral springs k_x and dashpots c_x and rotation springs k_θ and dashpots c_θ are the simplified horizontal soil reactions and vertical shear tractions, while the shear translation spring k_{bx} and dashpot c_{bx} and base rotation spring $k_{b\theta}$ and dashpot $c_{b\theta}$ are the simplified horizontal shearing force and the moment produced by the base of the shaft.

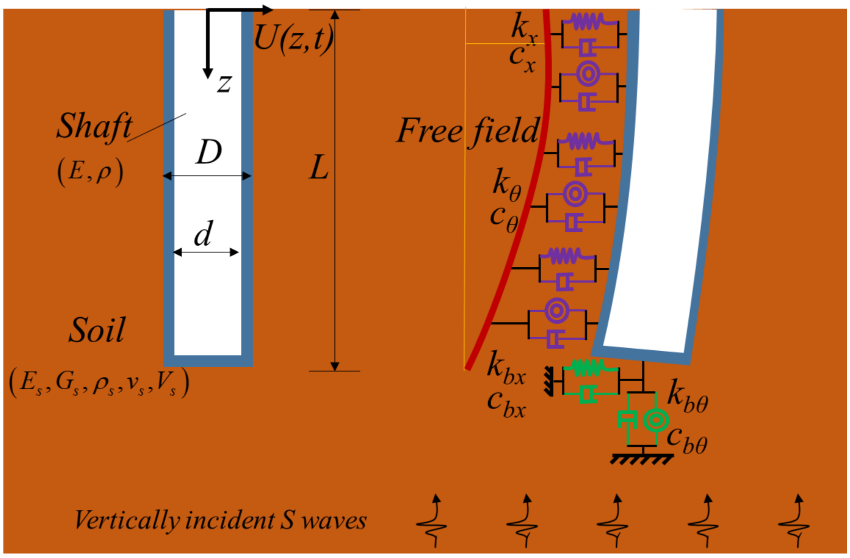

Figure 1. Schematic diagram of shaft under vertically incident S waves

3. Explicit representation of the model

3.1 Shaft kinematic responses equation

In order to establish the differential equation of the shaft kinematic responses, a shaft element is chosen from the physical model in Fig. 1. Fig. 2 illustrates the state of the beam element in a viscoelastic soil under the seismic loads. The main loads in horizontal direction are inertia force, the soil reaction force and the shear forces from the connecting shaft elements. The main moment loads are inertia moment, the moment form the soil vertical tractions, the soil reaction moment and the moment from the connecting shaft elements.

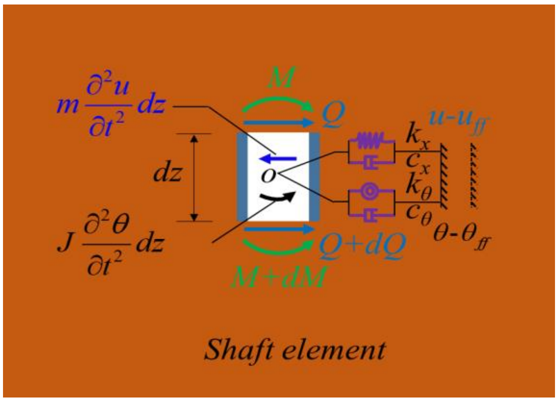

Figure 2. Schematic diagram of force acting on the shaft element under seismic motion

According to the dynamic equilibrium of the transverse forces and the dynamic moment equilibrium with respect to the central point O of the shaft element. The two differential governing equations for the shaft kinematic responses element in time domain can be expressed as

$$m\frac{\partial^2 u}{\partial t^2}dz + k_x(u-u_{ff})dz + c_x\frac{\partial(u-u_{ff})}{\partial t}dz = -dQ \tag{1}$$

$$J\frac{\partial^2 Q}{\partial t^2}dz + k_\theta(\theta-\theta_{ff})dz + c_\theta\frac{\partial(\theta-\theta_{ff})}{\partial t}dz = -dM + Qdz - \tau\frac{\pi D^2}{4}dz \tag{2}$$

where m is the mass of the shaft per length. J is the moment of inertia of the shaft per length. u (z, t) is the displacement of the shaft central. dz is the length of the shaft element. u_{ff} is the displacement of the free field along the depth The formulate of u_{ff} in frequency domain can be expressed as

$$u_{ff}\left(w,z\right) = u_{ff0}\cos\left(\frac{w}{V_s}z\right) \tag{3}$$

where u_{ff0} is the displacement of the soil surface. w is the circular frequency. z is the vertical coordinate starting from the top central of the shaft. θ_{ff} is the rotation angle of the free field. The formulate of θ_{ff} in frequency domain can be expressed as

$$\theta_{ff}\left(w,z\right) = -\frac{du_{ff}}{dz} = u_{ff0}\left(\frac{w}{V_s}\right)\sin\left(\frac{w}{V_s}z\right) \tag{4}$$

The bending moment M at the shaft cross section can be expressed as

$$M = -EI\frac{d^2u}{dz^2} \tag{5}$$

where I is the area moment of inertia of the shaft cross section. Q is the shear force at the shaft cross section

$$Q = -EI\frac{d^3u}{dz^3} \tag{6}$$

θ is the rotation angle of the shaft along the shaft depth

$$\theta = -\frac{du}{dz} \tag{7}$$

Submitting Eq. (1) in Eq. (2), one obtains the final dynamic equilibrium equation of the shaft element in frequency domain

$$EI\frac{d^4u}{dz^4} + (Jw^2 + iwc_\theta + k_\theta)\frac{d^2u}{dz^2} + (-mw^2 - iwc_x - k_x)u = \begin{pmatrix} (iwc_\theta + k_\theta)\dfrac{d^2u_{ff}}{dz^2} \\ +(-iwc_x - k_x)u_{ff} \end{pmatrix} \tag{8}$$

For express simplification, then name the equation coefficients as A_c, B_c, C_c and D_c.

$$A_c = EI \tag{9}$$

$$B_c = Jw^2 + iwc_\theta + k_\theta \tag{10}$$

$$C_c = -mw^2 - iwc_x - k_x \tag{11}$$

$$D_c = \left[\left(iwc_\theta + k_\theta\right)u_{ff0}\left(-\frac{w^2}{V_s^2}\right) + (-iwc_x - k_x)u_{ff0} \right]\cos(\frac{w}{V_s}z) \tag{12}$$

Then the dynamic equilibrium Eq. (8) could be simplified as

$$A_c\frac{d^4u}{dz^4} + B_c\frac{d^2u}{dz^2} + C_c u = D_c \tag{13}$$

The analytic solution the Eq. (8) in frequency domain can be expressed as

$$u(w,z) = C_1e^{r_1z} + C_2e^{r_2z} + C_3e^{r_3z} + C_4e^{r_4z} + E_c\cos(\frac{w}{V_s}z) \tag{14}$$

Where C_1, C_2, C_3 and C_4 are the four underdetermined parameters, which can be obtained through the boundary conditions. The parameters of r can be obtained as

$$\begin{cases} r_{1,2} = \pm\sqrt{\dfrac{-B_c - \sqrt{B_c^{\,2} - 4A_c C_c}}{2A_c}} \\ r_{3,4} = \pm\sqrt{\dfrac{-B_c + \sqrt{B_c^{\,2} - 4A_c C_c}}{2A_c}} \end{cases} \tag{15}$$

The coefficients of the specific solution E_c can be expressed as

$$E_c = \frac{D_c}{C_c - B_c(\frac{w}{V_s})^2 + (\frac{w}{V_s})^4} \tag{16}$$

The boundary condition about the shaft top is free at the shaft head and constraint at the bottom, then the boundary conditions can be formulated as follows

$$\begin{cases} M\big|_{z=0} = -EI\left.\dfrac{d^2u}{dz^2}\right|_{z=0} = 0 \\ Q\big|_{z=0} = -EI\left.\dfrac{d^3u}{dz^3}\right|_{z=0} = 0 \end{cases} \tag{17}$$

$$\begin{cases} M\big|_{z=l} = -EI\left.\dfrac{d^2u}{dz^2}\right|_{z=l} = \left(k_{b\theta} + iwc_{b\theta}\right)\left(-\left.\dfrac{du}{dz}\right|_{z=l}\right) \\ Q\big|_{z=l} = -EI\left.\dfrac{d^3u}{dz^3}\right|_{z=l} = \left(k_{bx} + iwc_{bx}\right)u\big|_{z=l} \end{cases} \tag{18}$$

Submitting the Eq. (14) into the Eq. (17) and (18), then the four undetermined parameters C_1, C_2, C_3 and C_4 can be obtained by the matric as follows

$$
\begin{bmatrix}
r_1^{\,2} & r_2^{\,2} & r_3^{\,2} & r_4^{\,2} \\
r_1^{\,3} & r_2^{\,3} & r_3^{\,3} & r_4^{\,3} \\
\left(r_1^{\,2}-\frac{k_r+iwc_r}{EI}\right)e^{r_1 l} & \left(r_2^{\,2}-\frac{k_r+iwc_r}{EI}\right)e^{r_1 l} & \left(r_3^{\,2}-\frac{k_r+iwc_r}{EI}\right)e^{r_1 l} & \left(r_4^{\,2}-\frac{k_r+iwc_r}{EI}\right)e^{r_1 l} \\
\left(r_1^{\,3}+\frac{k_h+iwc_h}{EI}\right)e^{r_1 l} & \left(r_2^{\,3}+\frac{k_h+iwc_h}{EI}\right)e^{r_1 l} & \left(r_3^{\,3}+\frac{k_h+iwc_h}{EI}\right)e^{r_1 l} & \left(r_4^{\,3}+\frac{k_h+iwc_h}{EI}\right)e^{r_1 l}
\end{bmatrix}
$$

$$
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}
=
\begin{bmatrix}
E_c\left(\frac{w^2}{V_s^{\,2}}\right) \\
0 \\
E_c\left[\left(\frac{k_r+iwc_r}{EI}\right)\left(\frac{-w}{V_s}\right)\sin\left(\frac{w}{V_s}L\right)-\left(\frac{-w^2}{V_s^{\,2}}\right)\cos\left(\frac{w}{V_s}L\right)\right] \\
-E_c\left[\left(\frac{k_h+iwc_h}{EI}\right)\cos\left(\frac{w}{V_s}L\right)+\left(\frac{w^3}{V_s^{\,3}}\right)\sin\left(\frac{w}{V_s}L\right)\right]
\end{bmatrix}
\tag{19}
$$

Finally, according to the Eqs. (5), (6), (7) and (14), the rotation angle, the bending moment and the shear force of the shaft along the depth in frequency domain can be obtained as follows

$$
\theta(w,z)=-\left[C_1r_1e^{r_1z}+C_2r_2e^{r_2z}+C_3r_3e^{r_3z}+C_4r_4e^{r_4z}+E_c\left(-\frac{w}{V_s}\right)\sin(\frac{w}{V_s}z)\right] \tag{20}
$$

$$
M(w,z)=-EI\left[C_1r_1^{\,2}e^{r_1z}+C_2r_2^{\,2}e^{r_2z}+C_3r_3^{\,2}e^{r_3z}+C_4r_4^{\,2}e^{r_4z}+E_c\left(-\frac{w^2}{V_s^{\,2}}\right)\cos(\frac{w}{V_s}z)\right] \tag{21}
$$

$$
Q(w,z)=-EI\left[C_1r_1^{\,3}e^{r_1z}+C_2r_2^{\,3}e^{r_2z}+C_3r_3^{\,3}e^{r_3z}+C_4r_4^{\,3}e^{r_4z}+E_c\left(\frac{w^3}{V_s^{\,3}}\right)\sin(\frac{w}{V_s}z)\right] \tag{22}
$$

3.2 Spring and dashpot coefficients

The four spring and dashpot coefficients are adopted from the coefficients which are proposed by Gazetas [9] and Gerolymos [6] and revised by Zhong [10]. They have done the calibration of the spring and dashpot coefficients with Novak, Varun and Wolf and verified that the coefficients perform well. This coefficients are related with the soil parameters and geometric parameters of shaft. The expression of the lateral horizontal spring coefficients k_x is as follows

$$
k_x=\left(\frac{I_{tw}\chi_{emb}-1}{L}\right)\frac{2.02E_sD}{\left(2-v_s\right)\left(1-v_s\right)} \tag{23}
$$

where I_{tw} is the horizontal embedment factor of a cylindrical shaft, χ_{ewb} is the dynamic coefficient.

$$I_{tw} = 1 + 0.21\left(\frac{L}{D}\right)^{0.5} + 1.43\left(\frac{L}{D}\right)^{0.8} + 0.3\left(\frac{L}{D}\right)^{1.3} \tag{24}$$

$$\chi_{emb} = 1 + \left(\frac{wD}{2V_s}\right)\left(\frac{L}{D}\right)\begin{bmatrix} \left(0.08 - -.0074\frac{L}{D}\right)\left(\frac{wD}{2V_s}\right)^2 - \left(0.31 - 0.0416\frac{L}{D}\right)\left(\frac{wD}{2V_s}\right) \\ -0.442\frac{L}{D} + 0.14 \end{bmatrix} \tag{25}$$

The expression of the lateral horizontal dashpot coefficients c_x is as follows

$$c_x = \left(2 + \frac{2.16}{1 - v_s}\right)\frac{E_s}{1 + v_s}\left(\frac{D}{2V_s}\right) \tag{26}$$

The expression of the lateral rotation spring coefficients k_θ is as follows

$$k_\theta = \frac{(\Gamma_w - 1)(1 - 0.3a_0)}{L}\frac{0.16E_sD^3}{(1 - v_s)(1 + v_s)} - \frac{1}{3}L^2\frac{(I_{tw}\chi_{emb} - 1)}{L}\frac{2.02E_sD}{(2 - v_s)(1 + v_s)} \tag{27}$$

where Γ_w is the rocking embedment factor of a cylindrical shaft.

$$\Gamma_w = 1 + 2.25\left(\frac{L}{D}\right)^{0.6} + 7.01\left(\frac{L}{D}\right)^{2.5} \tag{28}$$

The expression of the lateral rotation dashpot coefficients c_θ is as follows

$$c_\theta = \begin{pmatrix} \left[\left(\frac{2}{3} + \frac{0.72}{1 - v_s}\right)L^2 + \frac{1}{2}D^2\right]\frac{E_s}{1 + v_s}\left[0.25 + 0.65\left(\frac{L}{D}\right)^{-0.25}\sqrt{\frac{wD}{2V_s}}\right] \\ -\frac{1}{3}L^2\left(2 + \frac{2.16}{1\text{-}v_s}\right)\frac{E_s}{(1 + v_s)} \end{pmatrix}\left(\frac{D}{2V_s}\right) \tag{29}$$

The expression of the base shear spring coefficient k_{bx} is as follows

$$k_{bx} = \frac{2.02E_sD}{(2 - v_s)(1 - v_s)} \tag{30}$$

The expression of the base shear dashpot coefficient c_{bx} is as follows

$$c_{bx} = \frac{0.79 E_s D}{1+v_s}\left(\frac{D}{2V_s}\right) \tag{31}$$

The expression of the base shear spring coefficient $k_{b\theta}$ is as follows

$$k_{b\theta} = \frac{0.16 E_s D^3}{(1-v_s)(1-v_s)}\left(1-0.3\frac{wD}{2V_s}\right) \tag{32}$$

The expression of the base rotation dashpot coefficient $c_{b\theta}$ is as follows

$$c_{b\theta} = \frac{0.05 E_s D^3}{(1-v_s)(1+v_s)}\left(\frac{D}{2V_s}\right) \tag{33}$$

4. Solution technique

Inspect from the above analytical equations, the input parameters u_{ff0} should be in frequency domain. In order to know the transient seismic responses of the shaft, the input parameters u_{ff0} should be transformed into frequency domain which can be achieved by conducting FFT and the analytical results u and so on should be transformed into time domain which can be achieved by conducting iFFT. There are total three steps to obtain the dynamic responses of the shaft. The flowchart of the solve procedure is shown in Fig. 3.

The first step is to transform the time history of the input ground motion u_{ff0} into frequency domain through the FFT method. The ground motion u_{ff0} in frequency domain will be obtained with the corresponding frequency w. From the matric Eq. (19) the coefficients C_1, C_2, C_3 and C_4 can be obtained under the corresponding frequency w.

The second step is to obtain the shaft dynamic responses along the depth, such as: shaft displacement, rotational angle, bending moment and shear force in frequency domain through the Eq. (14), (20), (21) and (22). At the same time, the influence of ground motion frequency on the dynamic response of shaft can be observed.

The third step is to transform the shaft's frequency domain dynamic response parameters into time domain by implementing iFFT.

Repeat the above operations, then the shaft dynamic responses under stochastic earthquakes can be achieved with this simplified method. All this procedures can be implemented by MATLAB software efficiently.

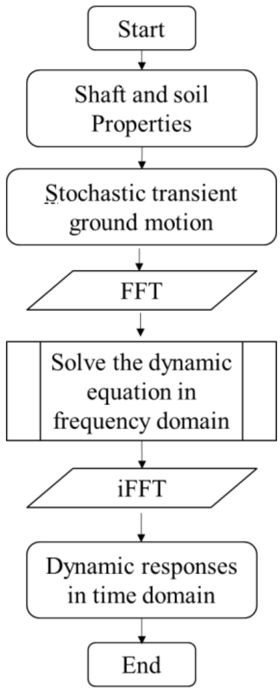

Figure 3. Flowchart of the solve procedure

5. Conclusion

Based on the theory of dynamic Winkle beam on elastic foundation, the shaft dynamic model is proposed and the closed-form solution for the dynamic responses of the shaft is established in frequency domain with explicit expression. A simplified model for calculating the seismic responses of the shaft under stochastic earthquakes is proposed and established. The seismic responses of shaft under stochastic earthquakes would be easily solved with the FFT and iFFT method between the frequency domain and the time domain. The simplified dynamic Winkle beam model could capture and properly reflect the shaft's dynamic kinematic responses, translation, rotational motion and bending curve along the depth with low computational cost compared with the three dimensional dynamic time history analysis.

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