Frictional contact analysis of functionally graded materials using smoothed finite element methods

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Abstract

In the paper, the smoothed finite element method (S-FEM) based on linear triangular elements is used to solve 2D solid contact problems of functionally graded materials. Both conforming and non-conforming contacts algorithms are developed using modified Coulomb friction contact models including tangential strength and normal adhesion. Based on smoothed Galerkin weak form, the system stiffness matrices are created using the formulation procedures of edge-based S-FEM (ES-FEM) and node-based S-FEM (NS-FEM), and the contact interface equations are discretized by contact point-pairs. Then these discretized system equations are converted into a form of linear complementarity problems (LCP), which can be further solved efficiently using the Lemke method. The singular value decomposition method is used to deal with the singularity of stiffness matrix in the procedure constructing the standard LCP, which can greatly improve the stability and accuracy of the numerical results. Numerical examples are presented to investigate the effects of functionally graded materials and comparisons have been made with analytical solutions and the standard FEM. The numerical results demonstrate that the strain energy solution of ES-FEM has higher convergence rate and accuracy compared with that of NS-FEM and FEM for functionally graded materials.

Key words: Smoothed finite element method; Contact problem; Linear complementarity problem; Strain energy; Functionally graded material

1. Introduction

Contact problems play an important role in many fields such as mechanical, civil engineering [1] and medicine. In fact, the use of functional graded materials (FGM) may become a critical issue for developing advanced lightweight structures, which meets the stringent requirements of high-tech fields. In many practical problems, the material of contact problems appears some particular physical properties changing with dimensions. For example, biological functionally gradient materials can be applied in medicine to achieve biological permanent repair and reconstruction of human hard tissue [4]. Therefore, it is necessary to study the contact problems of functionally graded materials.

Compared with solid mechanics, the geometric and material are discontinue at the contact interface, which made it difficult to solve by the analytical method, so the numerical methods are needed. In this work, we study the contact problems with functionally graded materials using the S-FEM. Based on smoothed Galerkin weak form, the system stiffness matrices are created using the formulation procedures of edge-based S-FEM (ES-FEM) and node-based S-FEM (NS-FEM), and the contact interface equations are discretized by contact point-pairs. However, the singularity of the system stiffness matrices makes the traditional solver for linear system of equation failed. We introduced the singular value decomposition method for solving linear system equations with singular system stiffness matrices. Through numerical simulations, it is demonstrated that the present algorithm is accurate and efficient for the contact problems of functionally graded materials.

2. Problem statement

2.1 Boundary value equations with contact

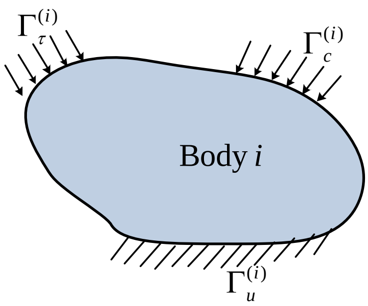


Figure 1. Configuration of the i th contact body

For solid body i as shown in Figure 1, the incremental forms from time t to $t+\Delta t$ of its static equilibrium equation, the constitutive equation and compatibility equation, the displacement increment, traction increment and contact traction increment boundary conditions are described, respectively, as follows

$$\begin{aligned}
&\mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma}^{(i)}+\boldsymbol{b}^{(i)}=0\\
&\boldsymbol{\sigma}^{(i)}=\bar{\mathbf{D}}^{(i)}\boldsymbol{\varepsilon}^{(i)}\\
&\boldsymbol{\varepsilon}^{(i)}=\mathbf{L}\mathbf{u}^{(i)}\\
&\mathbf{u}^{(i)}=\bar{\mathbf{u}}^{(i)},\text{on }\Gamma_u^{(i)}\\
&\mathbf{t}^{(i)}=\mathbf{L}_{\mathrm{n}}^{T}\boldsymbol{\sigma}^{(i)}=\bar{\mathbf{t}}^{(i)},\text{on }\Gamma_\tau^{(i)}\\
&\bar{\boldsymbol{\tau}}^{(i)}=\mathbf{L}_{\mathrm{n}}^{T}\boldsymbol{\sigma}^{(i)},\text{on }\Gamma_c^{(i)}
\end{aligned}\tag{1}$$

2.2 Modeling of contact interfaces

By using the slack vectors of residual strength and contact gap from Kuhn-Tucker conditions, we get the conformability equations as following [3]:

$$\begin{aligned}
&-\hat{\mathbf{M}}_c\hat{\boldsymbol{\tau}}-\hat{\mathbf{k}}+\hat{\boldsymbol{\lambda}}=0\\
&\hat{\mathbf{M}}_g\hat{\boldsymbol{\delta}}-\hat{\mathbf{g}}=0\\
&\hat{\boldsymbol{\lambda}}^{\mathrm{T}}\hat{\boldsymbol{\delta}}=0,\ \hat{\boldsymbol{\lambda}}\ge 0,\ \hat{\boldsymbol{\delta}}\ge 0
\end{aligned}\tag{2}$$

3. Smoothed Galerkin weak form

Based on the smoothing operator, we have the following smoothed Galerkin weak form with contact boundary:

$$\sum_{c=1}^{N_c^{(i)}} \int_{\bar{\Omega}_c^{(i)}} \delta \bar{\boldsymbol{\varepsilon}}_c^{(i)T} \bar{\mathbf{D}}^{(i)} \bar{\boldsymbol{\varepsilon}}_c^{(i)} d\Omega - \int_{\Omega^{(i)}} \delta \mathbf{u}^{(i)T} \boldsymbol{b}^{(i)} d\Omega \\ - \int_{\Gamma_t^{(i)}} \delta \mathbf{u}^{(i)T} \bar{\mathbf{t}}^{(i)} d\Gamma - \int_{\Gamma_c^{(i)}} \delta \mathbf{u}^{(i)T} \Theta^{(i)T} \boldsymbol{\tau}^{(i)} d\Gamma = 0, \tag{3}$$

4. Discretized form for equation

The physical properties of functional graded materials (FGM) change with dimensions. For example, the Young's modulus is not constant, so that the material constant matrix is different for each smoothing domain. For each edge-based smoothing domain, using the material constant of the midpoint of the edge to represent the material constant of the smoothing domain. Similarly, for each node-based smoothing domain, using the material constant of the node to represent the material constant of the smoothing domain. For 2D contact problem of FGM, the material constant matrix $\bar{\mathbf{D}}$ can be written as:

$$\bar{\mathbf{D}} = \left[c_{ij}(x,y) \right], x, y \in \Omega \tag{4}$$

where c_{ij} is the material constant, which change with dimensions.

Based on smoothed Galerkin weak form, the system stiffness matrices are created using the formulation procedures of edge-based S-FEM (ES-FEM) and node-based S-FEM (NS-FEM), and the contact interface equations are discretized by contact point-pairs. Then we get the following equations:

$$\begin{aligned} &\mathbf{KU - C\boldsymbol{\tau} = F} \\ &\mathbf{-M_c\boldsymbol{\tau} + \boldsymbol{\lambda} = k + M_c\boldsymbol{\tau}_t} \\ &\mathbf{M_g\boldsymbol{\delta} - GU = G_t} \\ &\boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{\delta} = 0,\ \boldsymbol{\lambda} \geq 0,\ \boldsymbol{\delta} \geq 0 \end{aligned} \tag{5}$$

From the Eq. (5) we have the following standard LCP, which can be very readily solved using the Lemke method.

$$\begin{cases} \boldsymbol{\lambda} = \mathbf{M}\boldsymbol{\delta} + \mathbf{q} \\ \tilde{\boldsymbol{\lambda}}^{\mathbf{T}}\tilde{\boldsymbol{\delta}} = 0,\ \tilde{\boldsymbol{\lambda}} \geq 0,\ \tilde{\boldsymbol{\delta}} \geq 0 \end{cases} \tag{6}$$

The contact tractions and the displacement of the entire domain can be calculated using:

$$\boldsymbol{\tau} = \bar{\mathbf{K}}^{-1}(\mathbf{M_g}\boldsymbol{\delta} - \mathbf{G_t} - \mathbf{GK^{-1}F}) \tag{7}$$

$$\mathbf{U} = \mathbf{K}^{-1}(\mathbf{F} + \mathbf{C}\boldsymbol{\tau}) \tag{8}$$

where $\mathbf{M} = \mathbf{M_c}\bar{\mathbf{K}}^{-1}\mathbf{M_g}$, $\mathbf{q} = -\mathbf{M_c}\bar{\mathbf{K}}^{-1}\left(\mathbf{G_t} + \mathbf{GK}^{-1}\mathbf{F}\right) + \mathbf{k} + \mathbf{M_c}\boldsymbol{\tau}_{\mathbf{t}}$, $\bar{\mathbf{K}} = \mathbf{GK}^{-1}\mathbf{C}$.

Note that the stiffness matrix $\mathbf{K}$ in Eq. (5) may be singular, because some contact bodies may be suspended namely without displacement boundaries or constraints for rigid body movement. Assume that the i th body is suspended, and then its stiffness matrix $\mathbf{K}^{(i)}$ may be singular for static analysis. Based on singular value decomposition method, the $m \times m$ real matrix $\mathbf{K}$ can be written as follows:

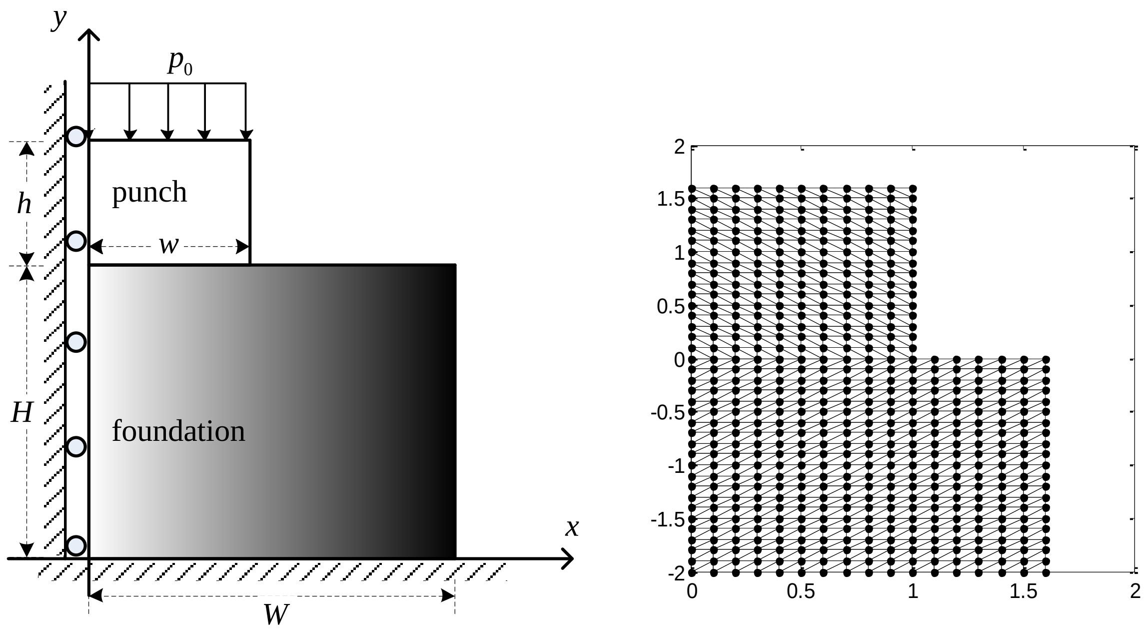
$$\mathbf{K} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}} \tag{9}$$

where $\mathbf{U}$ is an $m \times m$ orthogonal matrix, $\mathbf{\Sigma}$ is a $m \times m$ rectangular diagonal matrix with non-negative real numbers on the diagonal, $\mathbf{V}$ is an $m \times m$ orthogonal matrix and $\mathbf{V}^{\mathrm{T}}$ is the transpose of $\mathbf{V}$. The diagonal entries σ_i of $\mathbf{\Sigma}$ are known as the singular values of $\mathbf{K}$. Then the inverse of the stiffness matrix $\mathbf{K}$ can be written as

$$\mathbf{K}^{-1} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^{\mathrm{T}} \tag{10}$$

5. Numerical example

In this section, a flat punch on an elastic foundation subjected to a uniform load as shown in Figure 2. The material parameters of flat punch and foundation are $p_0 = 1\text{Mpa}, w = 1m, h = 1.6w, H = 2w, W = 1.6w$.



(a) (b)

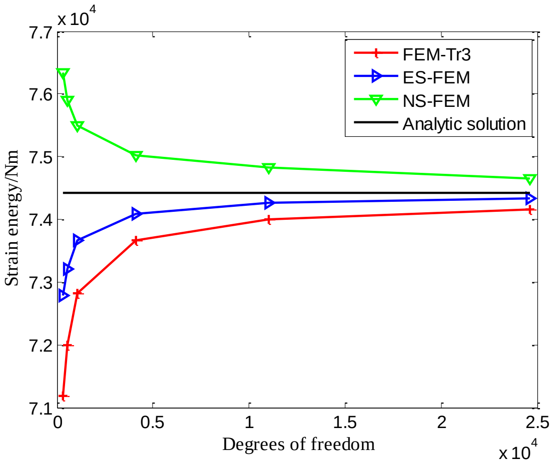
Figure 2. A flat punch on an elastic foundation subjected to a uniform load: (a) its half model with symmetric conditions imposed on the left; (b) discretized mesh model

5.1 Convergence of strain energy

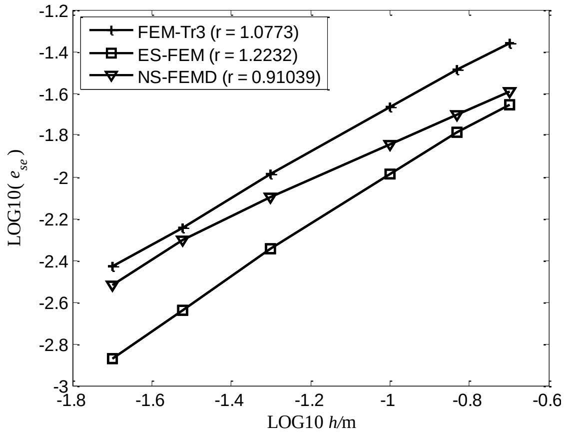
The Young's modulus and Poisson ratio of the punch and foundation are $E_p = 1\text{Gpa}$, $E_f = 10\text{Mpa}$ and $\nu_p = \nu_f = 0.3$. We use six different mesh models to discretize the problem domain listed in Table 1, and the convergence of strain energy is obtained as shown in Figure 3.

Table 1 The number of nodes, T3 elements and the density of boundary nodes for flat punch

No.	M1	M2	M3	M4	M5	M6	Ref.
Nodes	153	264	544	2046	5508	12312	48622
T3 elements	240	440	960	3840	10600	24000	48000
Boundary nodes density	0.2	0.15	0.1	0.05	0.03	0.02	0.01



(a)



(b)

Figure 3. Using different methods for flat punch on an elastic foundation: (a) convergence of strain energy;(b) convergence ratios of relative error of strain energy

From Figure 3 (a), it is obviously observed that the solution of the NS-FEM model converges from above to the reference solution, while the ES-FEM model converges from below to the reference solution with the increase of nodal degree of freedom. Moreover, the strain energy solutions obtained by NS-FEM and ES-FEM models converge linearly with the characteristic length of the mesh, shown in Figure 3 (b). The convergence rate of ES-FEM is larger than that of NS-FEM and FEM-Tr3, which is about 20% higher than that of FEM-T3. The numerical results demonstrate that strain energy solutions of ES-FEM have higher convergence rate and accuracy compared with that of NS-FEM and FEM.

5.2 Influence of functionally graded materials

Let the Poisson ratio $\nu_p = \nu_f = 0.3$, E_0 and E_R represent the Young's modulus at the center of the foundation and at the right respectively and $r = E_R / E_0$, $E_0 = 10\text{Mpa}$. Let E_p represent the Young's modulus of the punch and $\kappa = E_p / E_0 = 100$. Young modulus of foundation is defined as follows: $E_f(x) = E_0 + (E_R - E_0)x / W$. In the work, let $r = 0.5, 1, 2, 5, 10$, we study the effect on the trend of normal contact traction with different r based on ES-FEM.

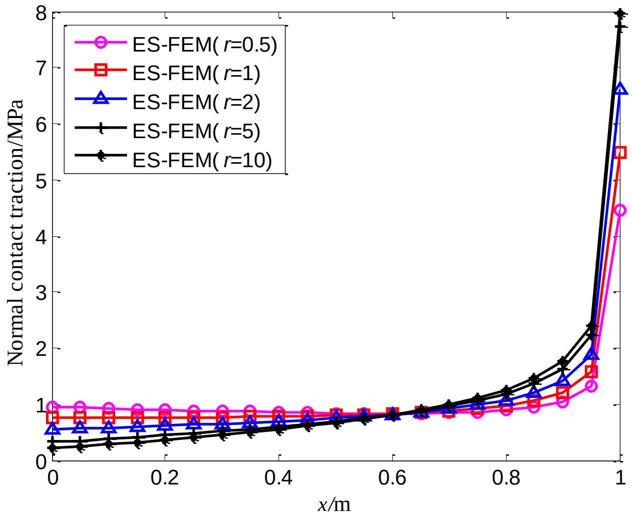


Figure 4. Normalized normal contact traction $\hat{\tau}_n$ in contact zone with different ratio r for flat punch on an elastic foundation

The Figure 4 shows that on the contact region from 0 to 0.6m, as r is increasing, the normal contact traction decreases. However, as the contact position is approaching to the right end of the punch, the contact traction increases with singularity. Furthermore, with the decreases of r the singularity of the right end decreases and the toughness of the elastic foundation can be maintained. The numerical example illustrates that the contact problems based on ES-FEM can be well solved for functional graded materials.

Conclusion

In the paper, we put forward conforming and non-conforming contacts algorithms using modified Coulomb friction contact models. Then according to S-FEM theory and contact point-pairs, discretized system equations are set up. Next, we use singular value decomposition method to solve the singularity of the stiffness matrix, through which the contact problems based on S-FEM can be well solved for functional graded materials. Through the intensive numerical simulations, we find that the presented algorithm is accurate and efficient for the contact problems of functionally graded materials.

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