Limit analysis of masonry structures based on fictitious associative-type contact interface laws

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Abstract

We illustrate an original method for the limit analysis of masonry structures modeled as assemblies of dry rigid blocks with Coulomb-type (non-associative) contact interface laws. The method resorts to a fictitious system characterized by cohesive-type contact interface laws that depend on the axial forces of the real block system. Two theorems establish the connection between the collapse state of the real (frictional) block assembly and that of the fictitious one. Hence, an alternative problem of mathematical programming is presented to evaluate the minimum collapse load multiplier. According to the proposed formulation, the complementarity condition is not introduced as constraint but is obtained as Karush-Khun-Tucker condition. Several numerical results concerning with masonry arches, portals and panels are provided to illustrate the application of the proposed approach, which is also validated through the comparison with some existing methods.

Keywords: **Masonry structure; Friction; Limit analysis; Mathematical programming.**

Introduction

Limit analysis provides an effective framework to study collapse load and failure mechanisms of the structures. Among the potential applications, those pertaining to block assemblies in presence of friction at the contact surfaces have received several attentions in the last decades because of the relevant practical implications. For instance, the collapse load estimation of rigid block systems interacting through frictional interfaces is of particular importance for the assessment of masonry structures. In this field, Baggio and Trovalusci [1] studied the limit analysis of no-tension and frictional three-dimensional discrete systems. In their study, the solution of the nonlinear programming problem is obtained by solving a preliminary linear programming problem that corresponds to a linearized limit analysis with dilatancy at the interfaces. Ferris and Tin-Loi [2] calculated the collapse loads of discrete rigid block systems with frictional contact interfaces by formulating a special constrained optimization problem and proposed an algorithm based on the relaxation of the complementarity constraint for its solution. The relaxation parameter is progressively reduced to zero through a succession of nonlinear sub-problems. Orduña and Lourenço [3] presented a model for the limit analysis of three-dimensional block assemblages interacting through frictional interfaces and included a proposal to take into account torsional failure modes. The model also accounted for limited compressive stresses at the interfaces. Gilbert and co-workers [4] illustrated an iterative procedure based on the successive solution of linear programming sub-problems. The method presented by these authors assumes fictitious values of cohesion and negative angles of friction, which are progressively relaxed toward zero. A finite-element-based approach has

been described by Mihai [5] for the limit analysis of planar systems formed by linear elastic bodies in non-penetrative contact with Coulomb friction.
In the present contribution, we illustrate a new method for the limit analysis of discrete systems formed by dry rigid blocks characterized by Coulomb-type (non-associative) contact interface laws [6]. The proposed method resorts to a discrete system with fictitious cohesive-type contact interface laws depending on the axial forces of the real block system. Once the connection between the collapse state of the fictitious block assembly and that of the real one is demonstrated, a new formulation of the mathematical programming problem intended to estimate the collapse load is proposed. In particular, the minimum collapse multiplier is here obtained by solving a nonlinear mathematical programming problem where the constraints include: (i) equilibrium conditions, (ii) kinematic conditions, and (iii) a further condition imposing that the collapse multiplier is kinematically admissible for the fictitious system with cohesive-type contact laws. In doing so, the classical complementarity condition is not introduced as constraint but is obtained as Karush-Khun-Tucker condition.

Proposed method for the limit analysis of masonry structures

An assembly of dry n_b blocks is considered. The constituent blocks are rigid and are allowed to slide over each other. Moreover, a Coulomb model is assumed to represent the frictional contact at the interfaces of the blocks. Contact forces and moments for the jth constituent block are defined as shown in Fig. 1.

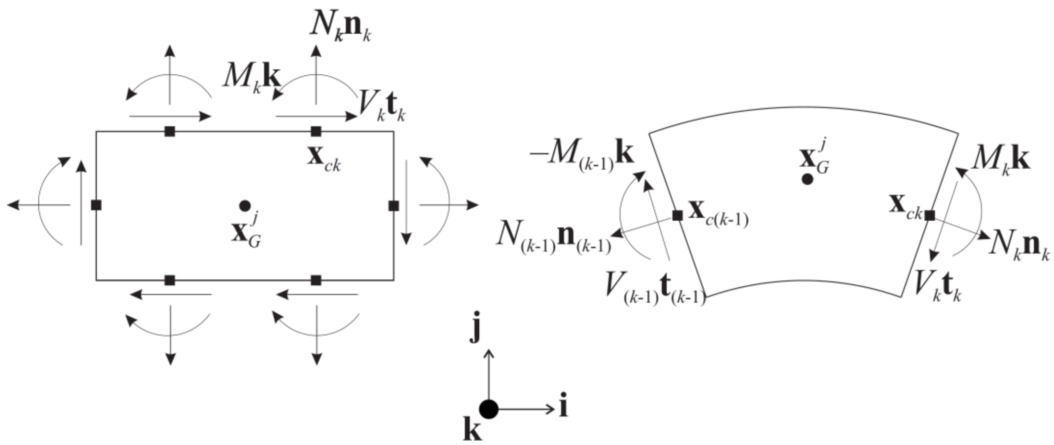

Figure 1. Contact forces and moments

The well-known equilibrium equations for the blocks are the following:

$$\begin{cases} \mathbf{f}_0^j + \alpha \bar{\mathbf{f}}^j + \sum\limits_{\mathbf{x}_{ck} \in \partial B^j} \left(V_k \mathbf{t}_k + N_k \mathbf{n}_k \right) = \mathbf{0} \\ \mathbf{x}_G^j \wedge \left(\mathbf{f}_0^j + \alpha \bar{\mathbf{f}}^j \right) + \sum\limits_{\mathbf{x}_{ck} \in \partial B^j} \left[\left(\mathbf{x}_{ck} - \mathbf{x}_G^j \right) \wedge \left(V_k \mathbf{t}_k + N_k \mathbf{n}_k \right) \pm M_k \mathbf{k} \right] = \mathbf{0} \end{cases}, \tag{1}$$

where j=1,…,n_b, $\mathbf{f}_0^j$ is the jth constant force vector acting on the jth block, $\bar{\mathbf{f}}^j$ is the jth base external force vector amplified by the load multiplier α. Moreover, N_k are the normal contact forces, V_k are the shear contact forces, M_k are the contact bending moments ($\mathbf{n}_k$, $\mathbf{t}_k$ and $\mathbf{k}$ are unit vectors). The kth resultant internal force or moment is applied on $\mathbf{x}_{ck}$, which is the kth contact interface point (with k=1,…,n_c) belonging to the boundary ∂B^j of the jth block whose center of mass is denoted as $\mathbf{x}_G^j$. Based on Eq. (1), the equilibrium of the structure can be expressed as follows [2]:

$$\mathbf{A}_f\mathbf{s}_f + \alpha\bar{\mathbf{f}} + \mathbf{f}_0 = \mathbf{A}\mathbf{s} + \mathbf{f}_0 = \mathbf{0}, \quad (2)$$

where $\alpha\bar{\mathbf{f}}$ and $\mathbf{f}_0$ are the vectors collecting the jth forces $\alpha\bar{\mathbf{f}}^j$ and $\mathbf{f}_0^j$, respectively, whereas $\mathbf{s}_f$ is the vector of the contact forces (internal forces and reactions). It is also introduced a vector $\mathbf{s}$ that includes the contact forces vector $\mathbf{s}_f$ and the load multiplier α, namely

$$\begin{aligned}\mathbf{s}^T &= \left\{\mathbf{s}_f^T \quad \alpha\right\} = \left\{\mathbf{s}_N^T \quad \mathbf{s}_V^T \quad \mathbf{s}_M^T \quad \alpha\right\} \\ &= \left\{ N_1 \quad \dots \quad N_{n_c} \quad V_1 \quad \dots \quad V_{n_c} \quad M_1 \quad \dots \quad M_{n_c} \quad \alpha \right\}\end{aligned}. \quad (3)$$

The following conditions must be fulfilled for such system at each contact interface:

$$\begin{cases}\mu N_k + V_k \le 0 \\ \mu N_k - V_k \le 0 \\ d_k N_k - M_k \le 0 \\ d_k N_k + M_k \le 0\end{cases}, \quad (4)$$

where μ is the static friction coefficient and $d_k>0$ is the maximum eccentricity of the resultant contact force at the kth contact surface. The conditions in Eq. (4) imply that the axial contact forces N_k must be negative or null.

It is now considered a fictitious (conjugate) block assembly, identical to the real one presented before but characterized by cohesive-type contact interface laws. The cohesive strengths in such fictitious system are taken equal to $-\mu\mathbf{s}_N$, thus depending on the axial contact forces N_k of the real system. Henceforth, S will denote the set that collects the statically admissible equilibrium states of the real block assembly whereas K will identify the set of the kinematically admissible displacement fields. The following theorems proved in Ref. [6] establish the connection between the collapse state of the real block system and that of the fictitious one:

- given any collapse state of the frictional block assembly, the collapse load multiplier α_c is always equal to the collapse load multiplier $\alpha_{as}(\mathbf{s}_N)$ of the block assembly with fictitious associative-type contact interface laws;
- given any statically admissible equilibrium state $\mathbf{s}$ of the frictional block assembly, if the load multiplier α is equal to any kinematically admissible load multiplier $\tilde{\alpha}_K$=($\mathbf{u}$,$\boldsymbol{\xi}$,$\mathbf{s}_N$) of the fictitious system, then $\mathbf{s}$ is a collapse state and $(\mathbf{u},\boldsymbol{\xi}) \in K$ is a collapse displacement field of the real system ($\mathbf{u}$ collects displacements and rotations of the blocks with respect to their centroids $\mathbf{x}_G^j$ whereas $\boldsymbol{\xi}$ collects relative displacements and rotations between the blocks).

Therefore, it is concluded that $\alpha=\alpha_{as}(\mathbf{s})$ if and only if $\mathbf{s}$ identifies a collapse state. Hence, the minimum collapse load multiplier α_c for the frictional (real) block assembly can be determined by solving the following mathematical programming problem [6]:

$$\begin{aligned}&\alpha_c = \min\left\{\alpha\right\} \\ &\begin{cases}\mathbf{s} \in S \\ \left(\mathbf{u},\boldsymbol{\xi}\right) \in K \\ \alpha = -\mathbf{u}^T\mathbf{f}_0 - \boldsymbol{\xi}^T\mathbf{N}_\mu\mathbf{s}_N\end{cases}\end{aligned}, \quad (5)$$

where N_μ=[μ**I** μ**I** **0** **0**]^T. Equation (5) provides an original approach for the limit analysis of frictional block assemblies. By solving the mathematical programming problem in Eq. (5), the minimum collapse load multiplier and the corresponding failure mode can be estimated. As far as the resolution technique is concerned, it is important to highlight that the admissible domain is not convex because of the last condition in Eq. (5). As a consequence, multiple local optima might exist and numerical resolution technique with global search capability is needed. In this work, the mathematical programming problem in Eq. (5) is solved by means of a genetic algorithm.

Numerical applications

An arch structure and two portals are first examined in order to illustrate the application of the proposed method. The *j*th base external force vector and the *j*th constant force vector are

$$\bar{\mathbf{f}}^j = \left\{ -W^j \quad 0 \quad 0 \right\}^T, \; \bar{\mathbf{f}}_0^j = \left\{ 0 \quad -W^j \quad 0 \right\}^T, \tag{6}$$

respectively, where W^j is the weight of the *j*th block. The weight per unit of volume of the blocks is 1.0 whereas the friction coefficient is 0.5. A unit-width slice of the structures is analyzed. Geometry and collapse mechanisms are shown in Figs. 2-4.

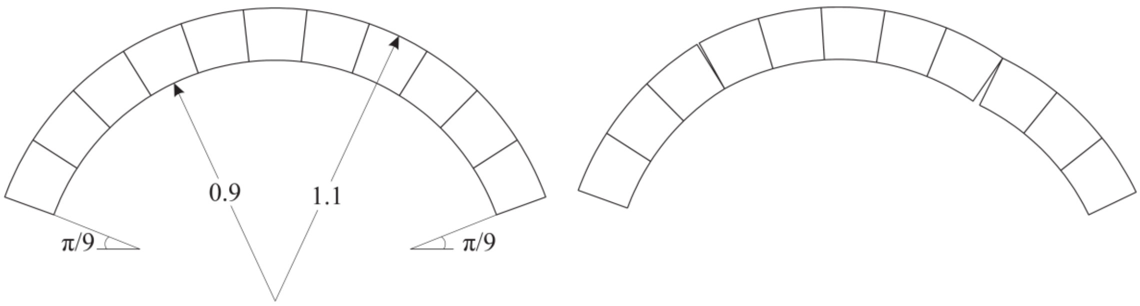

Figure 2. Geometry of the arch (left) and its collapse mechanism (right)

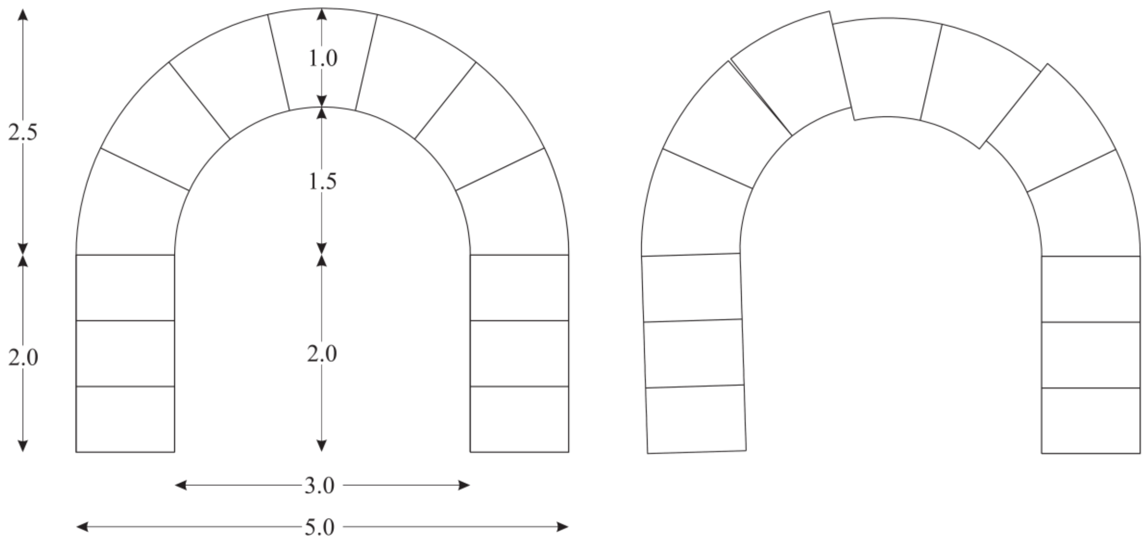

Figure 3. Geometry of the portal with constant thickness (left) and its collapse mechanism (right)

As shown in Fig. 2, the collapse of the masonry arch is based on a hinging-dominated mechanism (the corresponding collapse load multiplier is 0.63425). By counting the blocks from the left, the first hinge occurs at the intrados of the arch, between the third and the fourth voussoirs. The second hinge takes place at the extrados of the arch, between the eighth and the ninth blocks. Finally, a third hinge occurs at the right impost of the arch, and it is placed on the intrados. The left impost also slides along its base and moves toward the outside. Combined sliding and hinging collapse modes occur for the two masonry portals, as shown in Fig. 3 and Fig. 4 (the corresponding collapse load multipliers are 0.20628 and 0.30148, respectively). In both failure mechanisms, a hinge occurs at the base of the left pier, thus causing its counterclockwise rotation. Another hinge takes place on the intrados of the arches. The sliding collapse mode involves the keystone of the arches and some voussoirs adjacent to it.

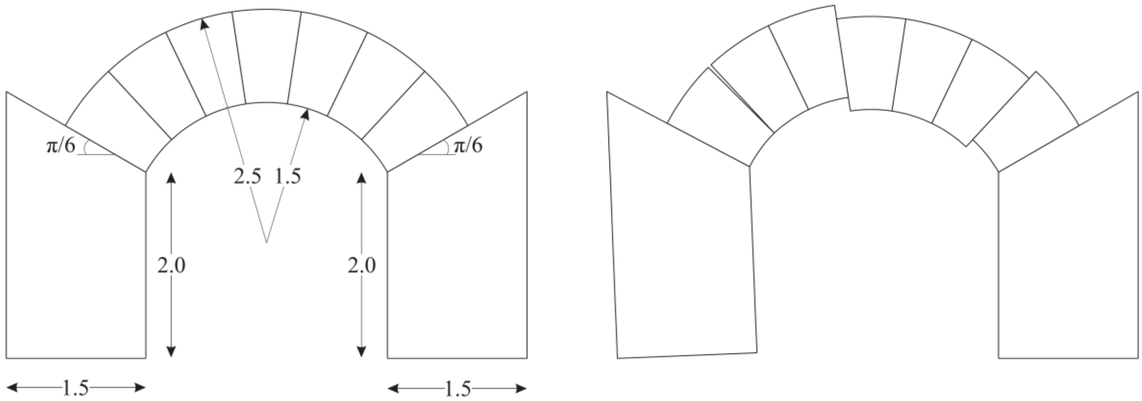

Figure 4. Geometry of the portal with large columns formed by a single block (left) and its collapse mechanism (right)

Two masonry panels are finally considered in order to demonstrate the correctness of the proposed method by comparing the corresponding collapse load multipliers with those estimated by Ferris and Tin-Loi [2], Gilbert et al. [4] and Mihai [5]. These examples are concerned with free standing walls supported on a rigid horizontal plane and subjected to in-plane forces applied to the centroid of each block. The full block size is 4×1.75 whereas the half block size is 2×1.75. The friction coefficient is 0.65. Each full block is subjected to a vertical body force (oriented downwards), which is calculated by assuming a weight equal to 1.0. Moreover, each full block is subjected to a unit horizontal live load (directed from left to right). One panel is formed by n_b=33 blocks whereas the second panel is formed by n_b=55 blocks. The collapse mechanisms of the examined panels are shown in Fig. 5. The corresponding collapse load multipliers are listed in Tab. 1, together with reference solutions reported in some existing studies. This comparison substantiates the correctness of the proposed approach.

Table 1. Collapse load multipliers of the considered walls

Wall	Ref. [2]	Ref. [4]	Ref. [5]	Proposed approach
n_b=33	0.63898	0.63982	0.63945	0.63911
n_b=55	0.55742	0.56262	0.55751	0.55749

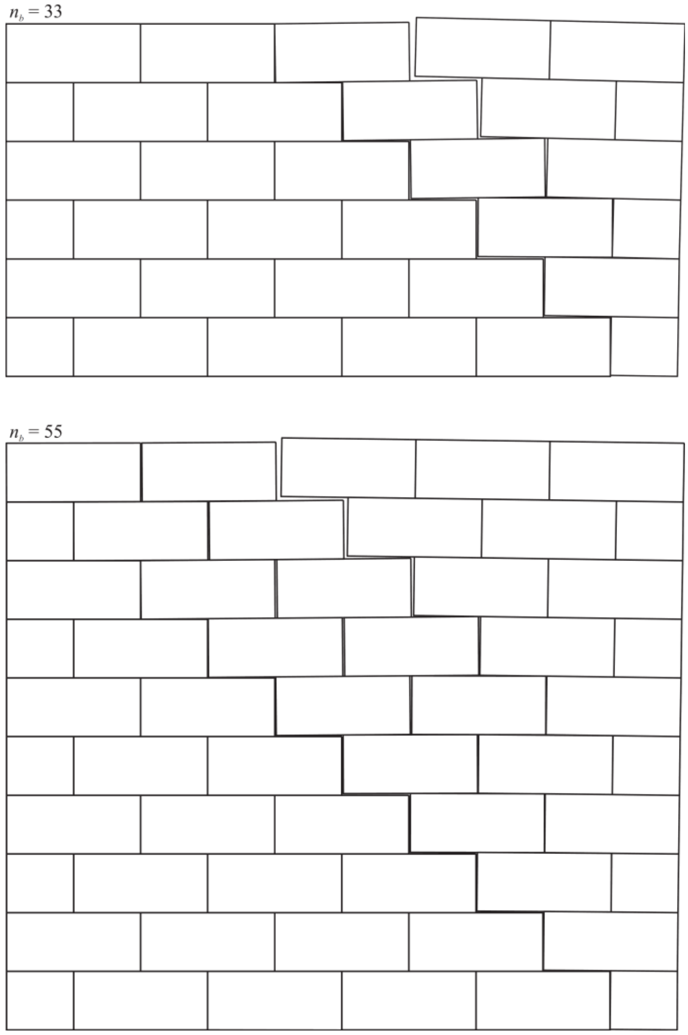

Figure 5. Collapse mechanisms of the considered walls

Conclusions

In the present work, we have illustrated an original strategy to address the limit analysis of frictional block assemblies by means of fictitious associative-type interfacial laws. Once the connection between the collapse state of the fictitious system and that of the real one has been highlighted, an original mathematical programming problem has been presented to estimate collapse load multiplier and failure mechanism. Herein, the introduction of the complementarity condition as constraint is not required because it is obtained as Karush-Khun-Tucker condition. Several numerical applications concerning with the limit analysis of masonry structures have been also included in order to demonstrate the application of the proposed approach and its correctness.

References

[1] Baggio, C. and Trovalusci, P. (1998) Limit analysis for no-tension and frictional three-dimensional discrete systems, *Mech Struct Mach* **26**(3), 287−304.
[2] Ferris, M. C. and Tin-Loi, F. (2001) Limit analysis of frictional block assemblies as a mathematical program with complementarity constraints, *Int J Mech Sci* **43**(1), 209−224.
[3] Orduña, A. and Lourenço P. B. (2005) Three-dimensional limit analysis of rigid blocks assemblages. PartI: torsion failure on friction interfaces and limit analysis formulation, *Int J Solids Struct* **42**(18-19), 5161−5180.

[4] Gilbert, M., Casapulla, C. and Ahmed, H. M. (2006) Limit analysis of masonry block structures with non-associative frictional joints using linear programming, *Comput Struct* **84**(13-14), 873−887.
[5] Mihai, L. A. (2010) A fixed-point approach to the limit load analysis of multibody structures with Coulomb friction, *Comput Struct* **88**(13−14), 859−869.
[6] Trentadue, F. and Quaranta, G. (2013) Limit analysis of frictional block assemblies by means of fictitious associative-type contact interface laws, *Int J Mech Sci* **70**, 140−145.