A class of novel tetrahedron elements with curved surfaces for three-dimensional solid mechanics problems with curved boundaries

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Abstract

Linear tetrahedral elements with four nodes (Te4) are currently the simplest and most widely used in finite element (FE) developed for solving 3D mechanics problems. However, the standard Te4 elements cannot be used to simulate accurately the 3D problems with curved boundaries, because of the flat surfaces of the standard Te4 elements. In this paper, we develop a set of new elements having curved surfaces to simulate the curved boundaries, by adding nodes to the standard Te4 elements. These novel elements include five-noded, six-noded, and seven-noded tetrahedron elements (Te5, Te6, and Te7). Based on the Te4 FE mesh, a hybrid mesh can be conveniently built for 3D problems with curved boundaries, in which the standard Te4 elements are used for the interior, and Te5, Te6, and Te7 elements are used for the curved boundaries. Compared with the standard FEM with Te4 elements, our mixing mesh can significantly improve the accuracy of the solution at the curved boundaries. Several solid mechanics problems are studied using hybrid meshes to validate the effectiveness of the present new elements.

Keywords: **finite element method; curved boundaries; five-noded, six-noded, seven-noded tetrahedron element**

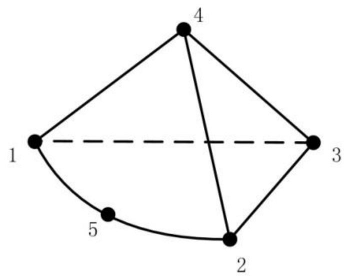
Introduction

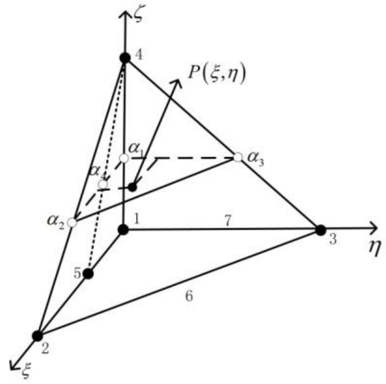
Common three-dimensional element of FEM is linear tetrahedral element with four nodes (Te4), which can automatically generate for complex geometries [1]. Due to its high efficiency, robustness and adaptability for complex geometries, the Te4 element is the most commonly used for general solid mechanics problems. However, the accuracy of FEMs using Te4 elements is poor in terms of stress, especially at the curved boundaries. Tetrahedral elements with ten nodes (Te10) [2], wedge elements with six nodes (W6) and hexahedron elements with eight nodes (H8) are proposed for this problem to obtain higher accuracy, but the computational cost is too large. Considering the above characteristics of the Te4, the Te10 and the higher order element, we propose a hybrid class of multi-node tetrahedral elements.

For the 3D problem domain with curved surfaces, some of the edges of tetrahedron boundary elements locate on the curved boundaries. If the edge is on the curved boundaries, we use a curved edge instead of the straight edge used in the standard Te4 element. Then we add a new node in the middle point of the curved edge to accurately simulate the changing trend. We analyze the type of the boundary elements, and put forward three kinds of new tetrahedron elements which are five-noded, six-noded and seven-noded tetrahedron (Te5, Te6, and Te7) elements.

The shape functions of the Te5, Te6 and Te7 elements for FEM

Then we construct the shape functions for the Te5, Te6 and Te7 elements. Figure 1(a) is a Te5 element that one additional node was added on the middle node of the curved edge. By using shape functions of the four-noded triangular (Tr4) element which can be found in [3], we can construct the shape function of a standard five-noded tetrahedron element in the natural system, which can be seen in Figure 1(b).





(a) (b)

Figure 1. (a) The five-noded tetrahedron element; (b) the five-noded tetrahedron element in natural system.

For any triangular $\Delta_{\alpha_1\alpha_2\alpha_3}$ paralleled to the triangular Δ_{123}, the displacement can be approximated using

$$u=\sum_{i=1}^{4}\phi_{\alpha_i}^{Tr4}u_{\alpha_i} \tag{1}$$

where the shape function of the Tr4 , $\Delta_{\alpha_1\alpha_2\alpha_3}$, can be written as

$$\begin{cases}\phi_{\alpha_1}^{Tr4}=\dfrac{2\left(\dfrac{\xi}{1-\zeta}\right)^2}{1-\left(\dfrac{\eta}{1-\zeta}\right)}-3\dfrac{\xi}{1-\zeta}+\left(1-\dfrac{\eta}{1-\zeta}\right)=\dfrac{2\xi^2}{(1-\zeta-\eta)(1-\zeta)}+\dfrac{1-3\xi-\zeta-\eta}{1-\zeta}\\ \phi_{\alpha_2}^{Tr4}=\dfrac{2\left(\dfrac{\xi}{1-\zeta}\right)^2}{1-\left(\dfrac{\eta}{1-\zeta}\right)}-\dfrac{\xi}{1-\zeta}=\dfrac{2\xi^2}{(1-\zeta-\eta)(1-\zeta)}-\dfrac{\xi}{1-\zeta}\\ \phi_{\alpha_3}^{Tr4}=\dfrac{\eta}{1-\zeta}\\ \phi_{\alpha_4}^{Tr4}=4\dfrac{\xi}{1-\zeta}-\dfrac{4\left(\dfrac{\xi}{1-\zeta}\right)^2}{1-\dfrac{\eta}{1-\zeta}}=\dfrac{4\xi}{1-\zeta}-\dfrac{4\xi^2}{(1-\zeta-\eta)(1-\zeta)}\end{cases} \tag{2}$$

Invoking the simple fact that

$$\frac{l_{4-\alpha_1}}{l_{4-1}}=\frac{l_{4-\alpha_2}}{l_{4-2}}=\frac{l_{4-\alpha_3}}{l_{4-3}}=\frac{l_{4-\alpha_5}}{l_{4-5}}=1-\zeta\ , \tag{3}$$

where l_{i-j} is the distance between two points *i* and *j*. So we have the relationships as listed:

$$u_{\alpha_i} = \zeta u_4 + (1-\zeta) u_i; \quad i = 1,2,3$$
$$u_{\alpha_i} = \zeta u_4 + (1-\zeta) u_{i+1}; \quad i = 4 \tag{4}$$

Therefore, the displacement in the standard five-noded tetrahedron element can be evaluated using the following formulation

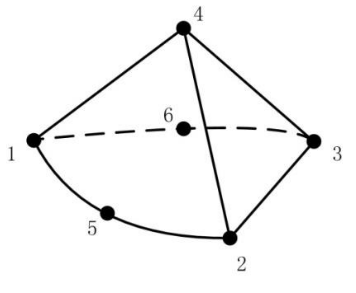
$$\begin{aligned}
u &= \sum_{i=1}^{4} \phi_{\alpha_i}^{Tr4} u_{\alpha_i} \\
&= \left(\zeta u_4 + (1-\zeta) u_1\right) \phi_{\alpha_1}^{Tr4} + \left(\zeta u_4 + (1-\zeta) u_2\right) \phi_{\alpha_2}^{Tr4} \\
&+ \left(\zeta u_4 + (1-\zeta) u_3\right) \phi_{\alpha_3}^{Tr4} + \left(\zeta u_4 + (1-\zeta) u_5\right) \phi_{\alpha_4}^{Tr4} \\
&= (1-\zeta) \phi_{\alpha_1}^{Tr4} u_1 + (1-\zeta) \phi_{\alpha_2}^{Tr4} u_2 + (1-\zeta) \phi_{\alpha_3}^{Tr4} u_3 + \zeta u_4 + (1-\zeta) \phi_{\alpha_4}^{Tr4} u_5 \\
&= N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 u_5
\end{aligned} \tag{5}$$

Substitute Eq.(2) into the above equation, the shape functions $N_i \left(i = 1,2,3,4,5\right)$ of the standard Te5 element can be formulated as

$$\begin{cases}
N_1 = \dfrac{2\xi^2}{(1-\zeta-\eta)} + 1 - 3\xi - \zeta - \eta \\
N_2 = \dfrac{2\xi^2}{(1-\zeta-\eta)} - \xi \\
N_3 = \eta \\
N_4 = \zeta \\
N_5 = 4\xi - \dfrac{4\xi^2}{(1-\zeta-\eta)}
\end{cases} . \tag{6}$$

where the parameters $\xi \in [0,1], \eta \in [0,1], \zeta \in [0,1]$.

Figure 2(a) is a Te6 element that two additional nodes were added on the middle node of each curved edge.



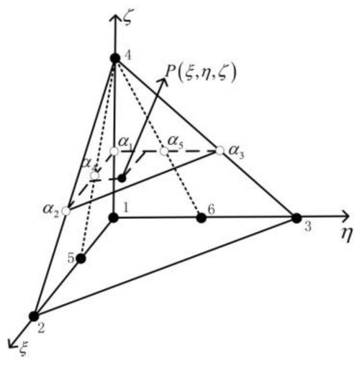


Figure 2. (a) The six-node tetrahedron element; (b) the standard six-node tetrahedron element in the natural system.

Similarly, the shape functions $N_i \left(i = 1,2,3,4,5,6\right)$ of the standard Te6 element, which can be seen in Figure 2(b), can be formulated as

$$\begin{cases} N_1 = (1-\zeta)\phi_{\alpha_1}^{Tr5} = \dfrac{(\xi+\eta+\zeta-1)(2\xi+2\eta+\zeta-1)}{(1-\zeta)} \\ N_2 = (1-\zeta)\phi_{\alpha_2}^{Tr5} = \dfrac{\xi(2\xi+2\eta+\zeta-1)}{(1-\zeta)} \\ N_3 = (1-\zeta)\phi_{\alpha_3}^{Tr5} = \dfrac{\eta(2\xi+2\eta+\zeta-1)}{(1-\zeta)} \\ N_4 = \zeta \\ N_5 = (1-\zeta)\phi_{\alpha_4}^{Tr5} = \dfrac{4\xi(1-\xi-\eta-\zeta)}{(1-\zeta)} \\ N_6 = (1-\zeta)\phi_{\alpha_5}^{Tr5} = \dfrac{4\eta(1-\xi-\eta-\zeta)}{(1-\zeta)} \end{cases} . \tag{7}$$

where the parameters $\xi \in [0,1], \eta \in [0,1], \zeta \in [0,1]$.

Figure 3(a) is a Te7 element that three additional nodes were added on the middle node of each curved edge.

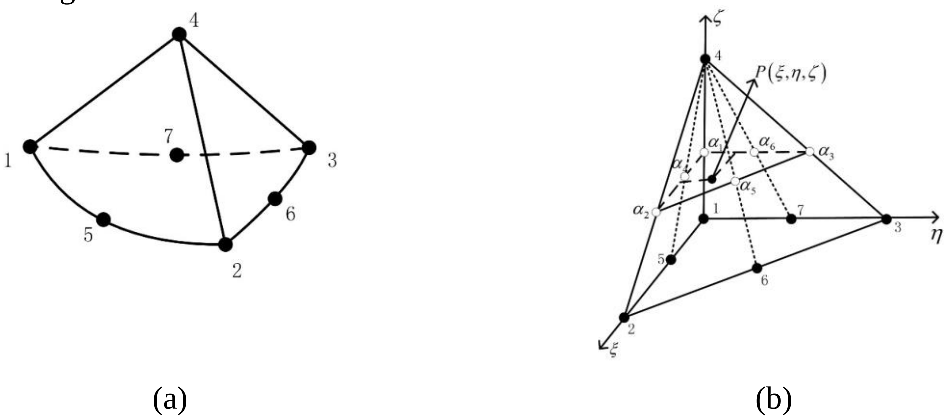


Figure 3. (a) The seven-node tetrahedron element; (b) the standard seven-node tetrahedron element in the natural system.

Similarly, the shape functions $N_i\,(i=1,2,3,4,5,6,7)$ of the Te7 element, which can be seen in Figure 3(b), can be formulated as

$$\begin{cases} N_1 = (1-\zeta)(2L_1-1)L_1 = (1-\xi-\eta-\zeta)\left(\dfrac{2(1-\xi-\eta-\zeta)}{1-\zeta}-1\right) \\ N_2 = (1-\zeta)(2L_2-1)L_2 = \xi\left(\dfrac{2\xi}{1-\zeta}-1\right) \\ N_3 = (1-\zeta)(2L_3-1)L_3 = \eta\left(\dfrac{2\eta}{1-\zeta}-1\right) \\ N_4 = \zeta \\ N_5 = 4(1-\zeta)L_1L_2 = \dfrac{4(1-\xi-\eta-\zeta)\xi}{1-\zeta} \\ N_6 = 4(1-\zeta)L_2L_3 = \dfrac{4\xi\eta}{1-\zeta} \\ N_7 = 4(1-\zeta)L_3L_1 = \dfrac{4(1-\xi-\eta-\zeta)\eta}{1-\zeta} \end{cases} . \tag{8}$$

where the parameters $\xi \in [0,1], \eta \in [0,1], \zeta \in [0,1]$.

Numerical simulation

The domain of the hollow sphere is defined as Ω = B(0, 2)/B(0, 1.0), which the origin O(0, 0, 0), inner radius a = 1.0m, and outer radius b = 2.0m. The hollow sphere is subjected to an internal pressure P=1 N/m^2 on the inner spherical surface. Because of the symmetric characteristics of the problem, only one-eighth of hollow sphere needs to be modeled as shown in Figure 1, and symmetric conditions are imposed on the symmetric planes.

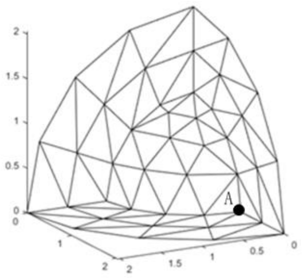


Figure 1. one-eighth of hollow sphere discretized using Te4 elements

Table1. Relative errors in displacement component *u* of the added nodes in the curved edges for the inner surface

Mesh	62 nodes	371 nodes	770 nodes	1482 nodes
FEM-Te4	0.2429	0.0532	0.0331	0.0140
FEM-HM	0.1796	0.0361	0.0246	0.0086

We use FEM-Te4 to represent the finite element method using Te4 elements and FEM-HM to represent the finite element method using a hybrid mesh with Te4, Te5, Te6 and Te7 elements. Table1 gives the relative errors in displacement component u of the added nodes in the curved edges for the inner surfaces using different elements and mesh sizes. The results show that the hybrid mesh with Te4, Te5, Te6 and Te7 elements can improve the accuracy of the displacement result on the curved boundaries, compared to the mesh with Te4 elements.

Figure 2 shows relative errors in radial stress σ_r of Point A (marked in Figure 1) against mesh sizes using different elements, which obtains the maximum radial stress easily observed in the analytical solution. It is clearly seen that the hybrid mesh with Te4, Te5, Te6 and Te7 elements stands out in the radial stress, compared the mesh with Te4 elements.

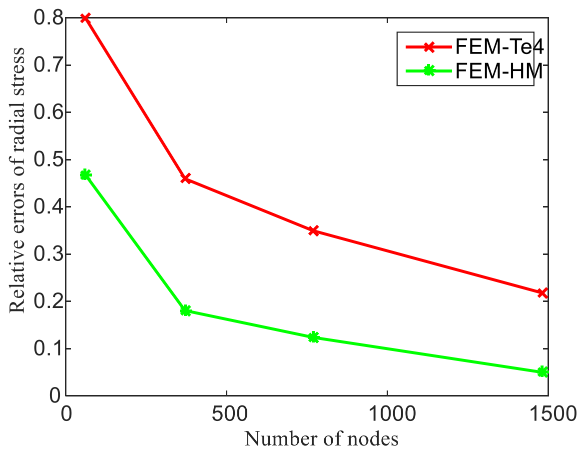


Figure 2. Relative errors of the radial stress of Point A against mesh sizes using different elements for 3D Lame problem.

Conclusions

In this paper, we present a novel hybrid mesh using Te5, Te6, and Te7 elements to accurately approximate the curved boundaries of problem domains. The hybrid mesh not only remains the advantages of the linear tetrahedral element, but also greatly improves the accuracy of the stress solution. Based on the shape functions of the standard Tr4 element, the standard Tr5 element and the standard Tr7 element, we formulate the shape functions for Te5, Te6, and Te7 elements separately. Through intensive numerical examples, it is concluded that our novel hybrid mesh with the multi-node tetrahedral element can simulate the curved boundaries efficiently and accurately.

References

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