An Optimal Eighth-Order Family of Iterative Methods For Multiple Roots

Fiza Zafar, Saima Akram*, Nusrat Yasmin

Centre for Advanced Studies in Pure and Applied Mathematics,
Bahauddin Zakariya University, Multan, Pakistan

*Corresponding author:saimaakram@bzu.edu.pk

Abstract

In this paper, we propose a new family of efficient and optimal iterative methods for multiple roots with known multiplicity $(m \geq 1)$. We use weight function approach involving one and two parameters to develop the new family. An extensive convergence analysis is discussed in order to demonstrate the optimal eighth-order convergence of the proposed scheme. Finally, numerical and dynamical tests are presented which confirm the theoretical results established in this paper and illustrate that the proposed family is efficient among the domain of multiple root finding methods.

Keywords: Nonlinear equations, multiple zeros, efficiency index, optimal iterative methods.

Introduction

The problem of solving nonlinear equation is recognized to be very old in history as many practical problems arising in nature are nonlinear. Various one-point and multi-point methods are presented to solve nonlinear equation or system of nonlinear equation [17, 18, 21]. The above cited methods are designed for the simple root of nonlinear equations but the behavior of these methods are not similar when dealing with multiple roots of nonlinear equations. The well known Newton's method with quadratic convergence for simple roots of nonlinear equations decays to first order when dealing with multiple roots of nonlinear equations. These problems lead to minor troubles such as greater computational cost and severe troubles such as no convergence at all. The prior knowledge of multiplicity of roots make it easier to deal with these difficulties. The anomalous behavior of the iterative methods while dealing with multiple roots is well known at least since 19th century when Schröder [20] developed a modification of classical Newton's method to preserve its 2nd order of convergence for multiple roots. The nonlinear equations with multiple roots commonly arise from different topics such as complex variables, fractional diffusion or image processing, applications to economics and statistics(Lẻvy distributions) etc. By knowing the practical nature of multiple root finders, various one-point and multi-point root solvers have been developed in recent past [1, 3, 4, 6-9, 11-15, 19, 24] but most of them are not optimal as defined by Kung and Traub [10] which states that an optimal without memory method can achieve its convergence order at most 2^n requiring $n+1$ evaluations of functions or derivatives. According to Ostrowski [17], if O is the convergence order of an iterative method and n is the total number of functional evaluations per iterative step then the index $E = O^{1/n}$ is known as efficiency index of an iterative method.

Sharma and Sharma [19] proposed the following optimal fourth order multiple root finder with known multiplicity m as follows:

$$y_n = x_n - \frac{2m}{m+2} \cdot \frac{f(x_n)}{f'(x_n)}, m > 1 \tag{1.1}$$

$$x_{n+1} = x_n - \frac{m}{8} \Phi(x_n) \frac{f(x_n)}{f'(x_n)},$$

where $\Phi(x_n) = \left\{ (m^3 - 4m + 8) - (m+2)^2 (\frac{m}{m+2})^m \frac{f'(x_n)}{f'(y_n)} \times 2(m-1)(m+2)(\frac{m}{m+2})^m \frac{f'(x_n)}{f'(y_n)} \right\}.$

Geum et al. in [7], presented a non-optimal family of two-point sixth-order methods to find multiple roots given as follows:

$$y_n = x_n - m \cdot \frac{f(x_n)}{f'(x_n)}, m > 1,$$

$$x_{n+1} = y_n - Q(r_n, s_n) \cdot \frac{f(y_n)}{f'(y_n)}, \tag{1.2}$$

where, $r_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}, s_n = \sqrt[m-1]{\frac{f'(y_n)}{f'(x_n)}}$ and $Q: \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in a neighborhood of $(0,0)$.

Following is a special case of their family:

$$y_n = x_n - m \cdot \frac{f(x_n)}{f'(x_n)}, n \geq 0, m > 1,$$

$$x_{n+1} = y_n - m\left[1 + 2(m-1)(r_n - s_n) - 4r_n s_n + s_n^2\right] \cdot \frac{f(y_n)}{f'(y_n)}. \tag{1.3}$$

Another non-optimal family of three-point sixth-order methods for multiple roots by Geum et al. [8], is given as follows:

$$y_n = x_n - m \cdot \frac{f(x_n)}{f'(x_n)}, m \geq 1,$$

$$w_n = y_n - m \cdot G(p_n) \cdot \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = w_n - m \cdot K(p_n, v_n,) \cdot \frac{f(x_n)}{f'(x_n)}, \tag{1.4}$$

where, $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$ and $v_n = \sqrt[m]{\frac{f(w_n)}{f(x_n)}}$. The weight functions $Q: \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of 0 and $K: \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in a neighborhood of $(0,0)$. Following is a special case of the family (1.4):

$$y_n = x_n - m \cdot \frac{f(x_n)}{f'(x_n)}, m \geq 1,$$

$$w_n = y_n - m \cdot \left[1 + p_n + 2p_n^2\right] \cdot \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = w_n - m \cdot \left[1 + p_n + 2p_n^2 + (1 + 2p_n)v_n\right] \cdot \frac{f(x_n)}{f'(x_n)}. \tag{1.5}$$

The families (1.2) and (1.4) require four functional evaluations to produce sixth order convergence with the efficiency index $6^{\frac{1}{4}} = 1.5650$ and therefore are not optimal in the sense of Kung-Traub's conjecture [10].

Recently, Behl et al. [2] have proposed a family of optimal eighth order iterative methods for multiple roots given as:

$$
\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, m \geq 1,\\
z_n &= y_n - u_n Q(h_n)\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= z_n - u_n t_n G(h_n, t_n)\frac{f(x_n)}{f'(x_n)},
\end{aligned} \tag{1.6}
$$

where, the weight functions $Q: \mathbb{C} \to \mathbb{C}$ and $G: \mathbb{C}^2 \to \mathbb{C}$ are analytic functions in a neighborhoods of (0) and (0,0), respectively, with $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $h_n = \frac{u_n}{a_1 + a_2 u_n}$ and $t_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$,

being a_1 and a_2 are complex non-zero free parameters.

We take particular case (27) for ($a_1 = 1$, $a_2 = 1$, $G_{02} = 0$) of the family by Behl et al. [2] and denote it by BM as follows:

$$
\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \left(m + 2h_n m + \frac{1}{2}h_n^2(4m + 2m)\right)\frac{f(x_n)}{f'(x_n)}u_n\\
x_{n+1} &= z_n - \left(m + mt_n + 3mh_n^2 + mh_n(2 + 4t_n + h_n)\right)\frac{f(x_n)}{f'(x_n)}u_n t_n.
\end{aligned} \tag{1.7}
$$

Most recently, second optimal eighth order scheme have been proposed by Zafar et al. [22], which is given as follows:

$$
\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, m \geq 1,\\
z_n &= y_n - m u_n H(u_n)\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= z_n - u_n t_n (B_1 + B_2 u_n) P(t_n) G(w_n)\frac{f(x_n)}{f'(x_n)},
\end{aligned} \tag{1.8}
$$

where $B_1, B_2 \in \mathbb{R}$are free parameters and the weight functions $H: \mathbb{C} \to \mathbb{C}$, $P: \mathbb{C} \to \mathbb{C}$ and $G: \mathbb{C} \to \mathbb{C}$ are analytic in the neighborhood of 0 with $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $t_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$ and $w_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$.

From the eighth order family of Zafar et al. [22], we consider the following special case denoted by ZM :

$$
\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - mu_n\left(6u_n^3 - u_n^2 + 2u_n + 1\right)\frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= z_n - mu_n t_n\left(1+2u_n\right)(1+t_n)(1+2w_n)\frac{f(x_n)}{f'(x_n)}.
\end{aligned} \tag{1.9}
$$

Optimal iterative methods are more significant than the non-optimal ones, regarding their efficiency and convergence speed. Therefore, there was a need to develop optimal eighth-order schemes for finding multiple zeros ($m>1$) as well as simple zeros ($m=1$) because of their better efficiencies and order of convergence [17], in addition optimal schemes require a small number of iterations to obtain desired accuracy as compare to fourth and sixth-order methods of Sharma and Geum [7, 8, 19]. In this paper, our main concern is to find the optimal iterative methods for multiple root μ with known multiplicity $m \in \mathbb{N}$ of a sufficiently differentiable nonlinear function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ where I is an open interval. We, in here, develop an optimal eighth order zero finder for multiple roots with known multiplicity $m \geq 1$. The beauty of the method lies in the fact that developed scheme is simple to implement with minimum possible number of functional evaluations. The family requires four functional evaluations to obtain eighth-order convergence with the efficiency index $8^{\frac{1}{4}} = 1.6817$.

The rest of the paper is organized as follows: In Section 2, we propose a new family of optimal eighth-order iterative methods to find multiple roots of nonlinear equations and discuss its convergence analysis. Some special cases are given in Section 3. In Section 4, numerical performance and comparison of the proposed schemes with the existing ones are given, dynamical analysis is given in section 5.Concluding remarks are given in Section 6.

Development of the scheme

In this section, we propose a new family of eighth-order method for a known multiplicity $m \geq 1$ of the desired multiple root as follows:

$$
\begin{aligned}
y_n &= x_n - m\cdot\frac{f(x_n)}{f'(x_n)}, n \geq 0, \\
z_n &= y_n - m\cdot t\cdot H(t)\cdot\frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= z_n - m\cdot t\cdot L(s,u)\cdot\frac{f(x_n)}{f'(x_n)},
\end{aligned}
$$

$$
\text{where} \quad t = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}, s = \sqrt[m]{\frac{f(z_n)}{f(y_n)}}, u = \sqrt[m]{\frac{f(z_n)}{f(x_n)}}, \tag{2.1}
$$

and the weight function $H: \mathbb{C} \to \mathbb{C}$ is analytic function in the neighborhood of 0 and weight function $L: \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in the neighborhood of $(0,0)$ and t, s and u are $one-to-m$ multiple -valued functions.

In the next theorem, it is demonstrated that the proposed scheme (2.1) achieves the optimal eighth order of convergence without increasing the number of functional evaluations.

Theorem 1 *Let* $x=\mu$ *(say) be a multiple zero with multiplicity* $m\geq 1$ *of an analytic function* $f:\mathbb{C}\to\mathbb{C}$ *in the region enclosing a multiple zero* μ *of* $f(x)$. *Then the family of iterative methods defined by (2.1) has eighth-order convergence when the following conditions are satisfied:*

$$H_0=1, H_1=2, H_2=-2, H_3=36, L_{00}=0, L_{10}=1, L_{01}=2, L_{11}=4, L_{20}=2. \tag{2.2}$$

Then the proposed scheme (2.1) satisfies the following error equations:

$$\begin{aligned} e_{n+1}=\frac{1}{24m^7}\{&c_1(c_1^2(11+m)-2mc_2)((677+108m+7m^2)c_1^4\\ &-24m(9+m)c_1^2c_2+12m^2c_2^2+12m^2c_1c_3)e_n^8\}+O(e_n^9), \end{aligned} \tag{2.3}$$

where $e_n=x_n-\mu$ *and* $c_k=\dfrac{m!}{(m+k)!}\dfrac{f^{(m+k)}(\mu)}{f^{(m)}(\mu)}, k=1,2,3,\cdots.$

Proof. Let $x=\mu$ be a multiple zero of $f(x)$. Expanding $f(x_n)$ and $f'(x_n)$ about $x=\mu$ by the Taylor's series expansion (with the help of computer algebra software Mathematica), we obtain

$$f(x_n)=\frac{f^{(m)}(\mu)}{m!}e_n^m\left(1+c_1e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+c_6e_n^6+c_7e_n^7+c_8e_n^8+O(e_n^9)\right), \tag{2.4}$$

and

$$\begin{aligned} f'(x_n)=\frac{f^{(m)}(\mu)}{m!}e_n^{m-1}&m+c_1(m+1)e_n+c_2(m+2)e_n^2+c_3(m+3)e_n^3+c_4(m+4)e_n^4\\ &+c_5(m+5)e_n^5+c_6(m+6)e_n^6+c_7(m+7)e_n^7+c_8(m+8)e_n^8+O(e_n^9), \end{aligned} \tag{2.5}$$

respectively. By using the above expressions (2.4) and (2.5) in the first substep of (2.1), we obtain

$$y_n-\mu=\frac{c_1e_n^2}{m}+\frac{(2c_2m-c_1^2(m+1))e_n^3}{m^2}+\sum_{k=0}^{4}G_ke_n^{k+4}+O(e_n^9), \tag{2.6}$$

where $G_k=G_k(m,c_1,c_2,\ldots,c_8)$ are expressed in terms of $m,c_1,c_2,\ldots,c_8$ where the two coefficients G_0 and G_1 can be explicitly written as

$G_0=\dfrac{1}{m^3}\{3c_3m^2+c_1^3(m+1)^2-c_1c_2m(3m+4)\}$ and

$G_1=-\dfrac{1}{m^4}\{c_1^4(m+1)^3-2c_2c_1^2m(2m^2+5m+3)+2c_3c_1m^2(2m+3)+2m^2(c_2^2(m+2)-2c_4m)\}$, etc.

With the help of Taylor's series expansion, we obtain

$$f(y_n)=f^{(m)}(\mu)e_n^{2m}\left[\frac{(\frac{c_1}{m})^m}{m!}+\frac{(2mc_2-(m+1)c_1^2)(\frac{c_1}{m})^m e_n}{c_1m!}+\sum_{k=0}^{4}\bar{G}_ke_n^{k+2}+O(e_n^9)\right]. \tag{2.7}$$

By using the expressions (2.4) and (2.7), we get

$$u=\frac{c_1e_n}{m}+\frac{(2mc_2-(m+2)c_1^2)e_n^2}{m^2}+\psi_1e_n^3+\psi_2e_n^4+\psi_3e_n^5+O(e_n^6), \tag{2.8}$$

where,

$$\psi_1=\frac{1}{2m^3}[c_1^3(2m^2+7m+7)+6c_3m^2-2c_1c_2m(3m+7)],$$
$$\psi_2=-\frac{1}{6m^4}[c_1^4(6m^3+29m^2+51m+34)-6c_2c_1^2m(4m^2+16m+17)+12c_1c_3m^2(2m+5)$$
$$+12m^2(c_2^2(m+3)-2c_4m)],$$
$$\psi_3=\frac{1}{24m^5}[-24m^3(c_2c_3(5m+17)-5c_5m)+12c_3c_1^2m^2(10m^2+43m+49)+12c_1m^2\{c_2^2(10m^2+47m+53)$$
$$-2c_4m(5m+13)\}-4c_2c_1^3m(30m^3+163m^2+306m+209)+c_1^5(24m^4+146m^3+355m^2+418m+209)]$$

Expanding Taylor series of $H(t)$ about 0 we have:

$$H(t)=H_0+H_1t+\frac{H_2}{2!}t^2+\frac{H_3}{3!}t^3+O(e_n{}^4) \tag{2.9}$$

where $H_j=H^j(0)$ for $0\le j\le 3$. Inserting the expressions (2.6)-(2.9) in the second substep of scheme (2.1), we have

$$z_n=\mu+\frac{-(1+H_0)c_1e_n^2}{m}-\frac{(1+H_1+m-H_0(3+m)c_1^2)+2(-1+H_0)mc_2)e_n^3}{m^2}$$
$$+\frac{1}{2m^3}\Big[(2+10H_1-H_2+4m+4H_1m+2m^2-H_0(13+11m+2m^2))c_1^3$$
$$+2m(-4-4H_1-3m+H_0(11+3m)c_1c_2-6(-1+H_0)m^2c_3)e_n^4\Big]+z_5e_n^5$$
$$+z_6e_n^6+z_7e_n^7+O(e_n^8).$$

By selecting $H_0=1$ and $H_1=2$ we obtained

$$z_n=\mu+\frac{(c_1^3(9-H_2+m)-2mc_1c_2)}{2m^3}e_n^4+z_5e_n^5+z_6e_n^6+z_7e_n^7+O(e_n^8), \tag{2.10}$$

where

$$z_5=-\frac{1}{6m^4}\{c_1^4(125+H_3+84m+7m^2-3H_2(7+3m)+6m(-3H_2+4(7+m))c_1^2c_2+12c_2^2m^2+12c_2c_1m),$$
$$z_6=\frac{1}{24m^5}\{1507+1850m+677m^2+46m^3+4H_3(9+4m)-6H2(59+53m+12m^2))c_1^5$$
$$-4m(925+8H_3+594m+53m^2-3H_2(53+21m)c_1^3c_2+12m^2(83-9H_2+13m)c_1^2c_3-168m^3c_2c_3$$
$$+12m^2c_1(115-12H2+17m)c_2^2-6mc_4)$$

and

$$z_7=-\{12c_1^2c_3m^2(36\beta+13m+11)+(37-168c_2c_3m^3+4c_1^3c_2m(96\beta^2+252\beta+53m^2+18(14\beta+5)m)$$
$$+12c_1m^2(c_2^2(48\beta+17m+19)-6c_4m)\}.$$

Now, again by using the Taylor's series expansion for (2.10), we have

$$f(z_n)=f^{(m)}(\mu)e_n^{4m}\frac{2^{-m}\left(\dfrac{c_1^3(9-H_2+m)-2mc_1c_2}{m^3}\right)^m}{m!}-\frac{\left(2^{-m}\left(\dfrac{c_1^3(9-H_2+m)-2mc_1c_2}{m^3}\right)^{m-1}\rho_0\right)}{3(m^3m!)}e_n \tag{2.11}$$

$$+\sum_{j=0}^{7}\overline{H}_j e_n^{j+1}+O(e_n^9),$$

where

$\rho_0 = c_1^4(125+H_3+84m+7m^2-3H_2(7+3m))c_1^4-6m(-3H2+4(7+m))c_1^2c_2+12m^2c_2^2+12c_3c_1m^2)$.

With the help of expressions (2.4) and (2.11), we have

$$s=\frac{c_1^2(9-H_2+m)-2mc_2}{2m^2}e_n^2+\rho_1 e_n^3+\rho_2 e_n^4+\rho_3 e_n^5+O(e_n^6), \tag{2.12}$$

where,

$\rho_1 = -(1/(6m^3))\{c_1{}^3(98+H^3+4m^2+54m-6H^2(3+m)-12m(9-H^2+m)c_1c_2+12m^2c_3\}$,

$\rho_2 = (1/(24m^4))899+1002m+313m^2+18m^3+4H_3(8+3m)-6H_2(43+33m+6m^2))c_1{}^4-12m(167+2H_3+87m+6m^2-H_2(33+10m)c_1{}^2c_2+24m^2(26-3H_2+3m)c_1c_3+12m^2(c_2{}^2(35-4H_2+3m)-6mc_4)$

and

$\rho_3 = -(1/(60m^5))[-4257-7270m-4455m^2-101m^3-48m^4-10H_3(37+30m+6m^2)+30H_2(60+75m+31m^2+4m^3)c_1{}^5+10m(1454+60H_3+1548m+21H_3m+454m^2+24m^3-18H_2(25+18m+3m^2)c_1{}^3c_2-30m^2(234+3H_3+118m+8m^2-2H2(24+7m)c_1{}^2c_3-60m^2c_1(141+2H_3+67m+4m^2-2H_2(15+4m)c_2{}^2+2(-17+2H_2-2m)mc_4)-120m^3(-25+3H_2-2m)c_2c_3+2mc_5\}+((1/(720m^6)))((102047+180H_2{}^2+204435m+187055m^2+81525m^3+14738m^4+600m^5+40H_3(389498m+214m^2+30m^3)-45H_2(1223+2030m+1353m^2+394m^3+40m^4))-30m(13629+22190m+12915m^2+2746m^3+120m^4+16H_3(83+64m+12m^2)-6H_2(1015+1209m+470m^2+56m^3))+120m^2(2063+2088m+589m^2+30m^{33}+H^3(88+30m)-18H_2+(36+25m+4m^2))+80m^2(2323+2348m+635m^2+30m^3+4H_3(289m)-3H_2(259+173m+26m^2))-2m(303+4H_3+149m+10m^2-9H_2(7+2m))-720m^3((393+6H_3+178m+10m^2-H_2(87+22m))]+(-42+5H_2-5m)mc_5)+20m^3\big((-473-8H^3-195m-10m^2+12H^2(9+2m)\big)c_2c_3+6m(65-8H2+5m)\,c_2+3m((71-9H2+5m)c_{10}mc_6$.

Since it is clear from (2.8) that u is of order e_n. Therefore, we can expand weight function $L(s,u)$ in the neighborhood of origin by Taylor's series expansion as follows:

$$L(s,u)=L_{00}+sL_{10}+uL_{01}+suL_{11}+\frac{s^2}{2!}L_{20}, \tag{2.13}$$

where $L_{i,j}=\dfrac{1}{i!\,j!}\dfrac{\partial^{i+j}}{\partial s^i \partial u^j}L(0,0)$. By using the expressions (2.4)-(2.13) in the proposed scheme (2.1), we have

$$e_{n+1}=M_2e_n^2+M_3e_n^3+M_4e_n^4+M_5e_n^5+M_6e_n^6+M_7e_n^7+O(e_n^8), \tag{2.14}$$

where the coefficients $M_i(2\le i\le 7)$ depends generally on m and the parameters $L_{i,j}$ .For obtaining at least fifth-order convergence, we have to choose $L_{00}=0, L_{10}=1$ and get

$$e_{n+1}=\frac{((-2+L_{01})c_1^2((-9+H_2-m)c_1^2+2mc_2)}{2m^4}e_n^5+\overline{M}_6e_n^6+\overline{M}_7e_n^7+O(e_n^8)$$

where the coefficients $\overline{M_\iota}(6\le i\le 7)$ depends generally on m and the parameters $L_{i,j}$. To obtain eighth order of convergence we choose the following values of parameters:

$$H_2 = -2, H_3 = 36, L_{00} = 0, L_{10} = 1, L_{01} = 2, L_{20} = 2, L_{11} = 4 \tag{2.15}$$

which leads us to the following error equation:

$$e_{n+1} = \frac{1}{24m^7}[c_1(c_1^2(11+m) - 2mc_2)((677+108m+7m^2)c_1^4 - 24m(9+m)c_1^2 c_2 + 12m^2 c_2^2 + 12m^2 c_1 c_3)]e_n^8 + O(e_n^9) \tag{2.16}$$

The above asymptotic error constant (2.16) reveals that the proposed scheme (2.1) reaches at optimal eighth-order convergence by using only four functional evaluations (using. $f(x_n), f'(x_n), f(y_n)$ and $f(z_n)$)per iteration.

Special Cases of Weight Functions

From Theorem 1, several choices of weight functions can be obtained, we have considered the following:

Case 1: The polynmial form of the weight function satisfying conditions (2.2) can be represented as:

$$\begin{aligned} H(t) &= 1 + 2t - t^2 + 6t^3 \\ L(s,u) &= s + 2u + 4su + s^2 \end{aligned} \tag{3.1}$$

A particular iterative method related to (3.1) is given by:

SM-1:

$$\begin{aligned} y_n &= x_n - m \cdot \frac{f(x_n)}{f'(x_n)}, n \ge 0, \\ z_n &= y_n - m \cdot t \cdot (1 + 2t - t^2 + 6t^3) \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - m \cdot t \cdot (s + s^2 + 2u + 4su) \cdot \frac{f(x_n)}{f'(x_n)} \\ \text{where } t &= \sqrt[m]{\frac{f(y_n)}{f(x_n)}}, s = \sqrt[m]{\frac{f(z_n)}{f(y_n)}}, u = \sqrt[m]{\frac{f(z_n)}{f(x_n)}} \end{aligned} \tag{3.2}$$

Case 2: The second suggested form of the weight functions in which $k_f(t)$ is constructed using rational weight function satisfying conditions (2.2) is given by:

$$\begin{aligned} H(t) &= \frac{1 + 8t + 11t^2}{1 + 6t} \\ L(s,u) &= s + 2u + 4su + s^2 \end{aligned} \tag{3.3}$$

The corresponding iterative method (3.3)can be presented as:

SM-2:

$$y_n = x_n - m\cdot\frac{f(x_n)}{f^{'}(x_n)}, n\geq 0,$$

$$z_n = y_n - m\cdot t\cdot(\frac{1+8t+11t^2}{1+6t})\frac{f(x_n)}{f^{'}(x_n)},$$

$$x_{n+1} = z_n - m\cdot t\cdot(s+s^2+2u+4su)\cdot\frac{f(x_n)}{f^{'}(x_n)},$$

where $$t=\sqrt[m]{\frac{f(y_n)}{f(x_n)}}, s=\sqrt[m]{\frac{f(z_n)}{f(y_n)}}, u=\sqrt[m]{\frac{f(z_n)}{f(x_n)}} \tag{3.4}$$

Case 3: The third suggested form of the weight function in which $K_f(t)$ is constructed using trigonometric weight satisfying conditions (2.2) is given by:

$$H(t) = \frac{5+18t}{5+18t-11t^2}$$

$$L(s,u) = s+2u+4su+s^2 \tag{3.5}$$

The corresponding iterative method obtained using (3.5) is given by:

SM-3:

$$y_n = x_n - m\cdot\frac{f(x_n)}{f^{'}(x_n)}, n\geq 0,$$

$$z_n = y_n - m.t.\left(\frac{5+18t}{5+18t-11t^2}\right)\frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - m\cdot t\cdot(s+s^2+2u+4su)\cdot\frac{f(x_n)}{f^{'}(x_n)},$$

where $$t=\sqrt[m]{\frac{f(y_n)}{f(x_n)}}, s=\sqrt[m]{\frac{f(z_n)}{f(y_n)}}, u=\sqrt[m]{\frac{f(z_n)}{f(x_n)}}. \tag{3.6}$$

Numerical tests

In this section, we show the performance of the presented iterative family (2.1) by carrying out some numerical tests and comparing the results with existing method for multiple roots. All the numerical computations have been performed in Maple 16 programming package using 1000 significant digits of minimum number of precision. In that case μ is not exact, it is replaced by a more accurate value which has more number of significant digits than the assigned precision. The test functions along with their roots μ and multiplicity m are listed in Table 1 [16]. The proposed methods SM-1 (3.2), SM-2 (3.4) and SM-3(3.6) are compared with the methods of Geum et al. given in (1.3) and (1.5) denoted by GKM-1 and GKM-2 and with method of Bhel given in (1.7) denoted by BM and Zafar et. al method given in (1.9) denoted by ZM respectively. Tables 2-8 display the errors of approximations to the sought zeros ($|x_n-\mu|$) produced by different methods at the first three iterations, where $E(-i)$ denotes $E\times 10^{-i}$. The initial approximation x_0 for each test function and computational order of convergence (COC) is also included in these tables, which is computed by the following expression [23]:

$$COC \approx \frac{\log|(x_{k+1}-\mu)/(x_k-\mu)|}{\log|(x_k-\mu)/(x_{k-1}-\mu)|}.$$

It is observed that, the performance of new method SM-2 is same as method of BM for function f_1 and better than method of ZM for function f_2. The newly developed schemes SM-1, SM-2 and SM-3 are not only convergent but also their speed of convergence is better than methods of GKM-1 and GKM-2. On the other hand methods of ZM and BM show divergence for function f_3. For f_4, f_5, f_6 and f_7 the newly developed schemes newly developed schemes SM-1, SM-2 and SM-3 are comparable with methods of ZM and BM. Hence, we conclude that the proposed family is comparable and robust among existing methods for multiple roots.

Table 1: Test functions

Test Functions	Exact root μ	Multiplicity m
$f_1(x)=(cos(\frac{\pi x}{2})+x^2-\pi)^5$	2.034724896...	5
$f_2(x)=(e^x+x-20)^2$	2.842438953...	2
$f_3(x)=(\ln x+\sqrt{(x^4+1)}-2)^9$	1.222813963...	9
$f_4(x)=(cosx-x)^3$	0.7390851332...	3
$f_5(x)=((x-1)^3-1)^{50}$	2.0	50
$f_6(x)=(x^3+4x^2-10)^6$	1.365230013...	6
$f_7(x)=(8xe^{-x^2}-2x-3)^8$	-1.7903531791...	8

Table 2: Comparison of different methods for multiple roots

$f_1(x)$,x_0=2.5							
	GKM-1	GKM-2	SM-1	SM-2	SM-3	ZM	BM
$\|x_1-\mu\|$	6.83(-4)	1.11(-3)	2.15(-4)	1.87(-4)	2.03(-4)	1.52(-4)	1.84(-4)
$\|x_2-\mu\|$	3.42(-14)	2.53(-18)	2.37(-29)	3.53(-30)	1.25(-29)	9.69(-31)	2.89(-30)
$\|x_3-\mu\|$	2.13(-55)	3.58(-106)	5.28(-299)	5.71(-236)	2.53(-231)	2.56(-240)	1.05(-236)
COC	4.00	6.00	8.00	8.00	8.00	8.00	8.00

Table 3: Comparison of different methods for multiple roots

$f_2(x)$,x_0=3.0							
	GKM-1	GKM-2	SM-1	SM-2	SM-3	ZM	BM
$\|x_1-\mu\|$	1.18(-7)	5.27(-6)	2.33(-7)	1.21(-7)	1.90(-7)	1.40(-7)	1.16(-7)
$\|x_2-\mu\|$	2.62(-37)	1.15(-32)	1.30(-53)	2.21(-56)	1.99(-54)	1.30(-55)	1.57(-56)
$\|x_3-\mu\|$	3.07(-221)	1.25(-192)	1.19(-423)	2.67(-446)	2.87(-430)	7.37(-440)	1.73(-447)
COC	4.00	6.00	8.00	8.00	8.00	8.00	8.00

Table 4: Comparison of different methods for multiple roots

$f_3(x)$,x_0=3.0							
	GKM-1	GKM-2	SM-1	SM-2	SM-3	ZM	BM
$\lvert x_1-\mu\rvert$	5.50(-1)	4.29(-2)	1.81(-2)	1.75(-2)	1.79(-2)	*	*
$\lvert x_2-\mu\rvert$	3.99(-7)	8.77(-10)	2.82(-15)	9.58(-16)	2.04(-15)	*	*
$\lvert x_3-\mu\rvert$	1.13(-27)	7.51(-56)	2.06(-117)	8.21(-122)	6.49(-119)	*	*
COC	4.00	6.00	8.00	8.00	8.00	*	*

"*" stands for divergence

Table 5: Comparison of different methods for multiple roots

$f_4(x)$,x_0=1.0							
	GKM-1	GKM-2	SM-1	SM-2	SM-3	ZM	BM
$\lvert x_1-\mu\rvert$	2.77(-4)	2.55(-5)	6.78(-8)	5.45(-8)	6.29(-8)	4.90(-8)	5.15(-8)
$\lvert x_2-\mu\rvert$	3.28(-14)	6.83(-36)	7.95(-60)	8.55(-61)	3.83(-60)	4.06(-61)	4.91(-61)
$\lvert x_3-\mu\rvert$	5.86(-49)	2.51(-213)	2.82(-475)	3.11(-483)	7.18(-478)	8.99(-486)	3.36(-485)
COC	3.50	6.00	8.00	8.00	8.00	7.99	7.99

Table 6: Comparison of different methods for multiple roots

$f_5(x)$,x_0=2.1							
	GKM-1	GKM-2	SM-1	SM-2	SM-3	ZM	BM
$\lvert x_1-\mu\rvert$	7.68(-5)	1.12(-5)	7.58(-7)	4.85(-7)	6.52(-7)	4.77(-7)	4.65(-7)
$\lvert x_2-\mu\rvert$	3.49(-17)	5.33(-29)	3.70(-47)	4.10(-49)	8.82(-48)	5.66(-49)	2.72(-49)
$\lvert x_3-\mu\rvert$	1.46(-66)	6.11(-169)	2.82(-369)	1.06(-385)	9.93(-375)	2.22(-384)	3.79(-387)
COC	3.99	6.00	8.00	8.00	8.00	7.99	7.99

Table 7: Comparison of different methods for multiple roots

$f_6(x)$,x_0=3.0							
	GKM-1	GKM-2	SM-1	SM-2	SM-3	ZM	BM
$\lvert x_1-\mu\rvert$	5.44(-2)	1.01(-1)	5.40(-2)	5.30(-2)	5.36(-2)	4.36(-2)	5.39(-2)
$\lvert x_2-\mu\rvert$	7.40(-7)	5.37(-7)	1.10(-10)	8.60(-11)	8.60(-11)	1.36(-11)	4.92(-11)
$\lvert x_3-\mu\rvert$	3.54(-26)	1.86(-38)	5.28(-80)	576(-81)	5.76(-81)	1.80(-87)	3.14(-83)
COC	3.97	5.96	8.00	7.98	7.97	7.97	7.97

Table 8: Comparison of different methods for multiple roots

$f_7(x)$,x_0=-1.2							
	GKM-1	GKM-2	SM-1	SM-2	SM-3	ZM	BM
$\lvert x_1-\mu\rvert$	2.65(-3)	2.15(-3)	4.38(-7)	4.24(-4)	4.32(-7)	3.41(-4)	4.26(-4)
$\lvert x_2-\mu\rvert$	7.24(-12)	9.63(-17)	4.44(-27)	1.11(-27)	3.11(-27)	3.58(-28)	1.14(-27)
$\lvert x_3-\mu\rvert$	4.05(-46)	7.81(-97)	4.97(-211)	2.55(-216)	2.28(-212)	5.27(-220)	3.06(-216)
COC	4.00	6.00	8.00	8.00	8.00	7.99	7.99

Dynamical analysis

Regarding the stability comparison, we use the routines presented in [5] for plotting the dynamical planes corresponding to each method (SM-1,SM-2, S-M3, BM and ZM) for the non-linear functions $f_1, f_2, f_3, f_4, f_5, f_6, f_7$. For this, we define a mesh of 400 $\times$ 400 points, as each point of the mesh is an initial guess for the analyzed method on the specific nonlinear function. If the sequence of iteration method reaches (closer than 10^{-3}) the multiple root in less than 80 iterations, then this point is painted in orange color; if the iterate converges to another thing (strange fixed points, cycles, etc.) then the point is painted black. The multiple root is represented in the different figures by a white star.

We observe from Figures 1–7 that the only basin of attraction is that of the multiple root (that is, the set of initial points converging to it fills all the plotted region of the complex plane), plotted in orange in the figures; although in general, convergence to other roots, divergence or even convergence to other fixed points that are not roots of the non-linear function (known as strange fixed points), can appear. We see in the figures, that the orange region is more bigger and brighter for the proposed schemes SM1, SM2 and SM3 than the regions of methods BM and ZM for all examples, that confirms their stability and fast convergence speed.

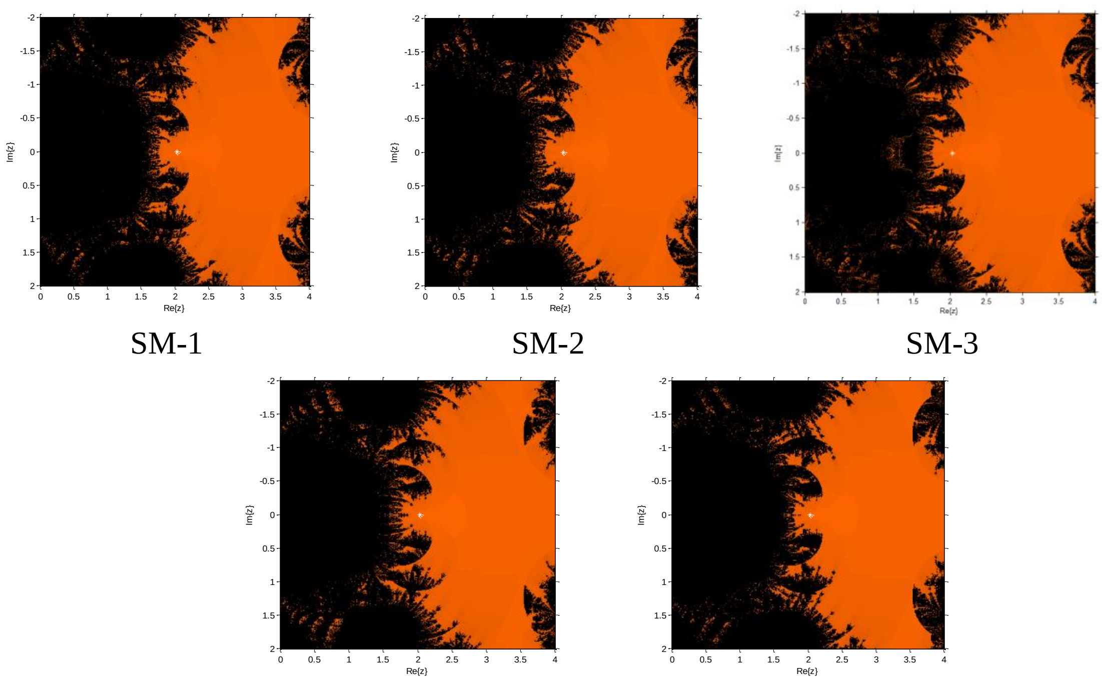


Fig. 1: Basins of attraction of different methods for f_1

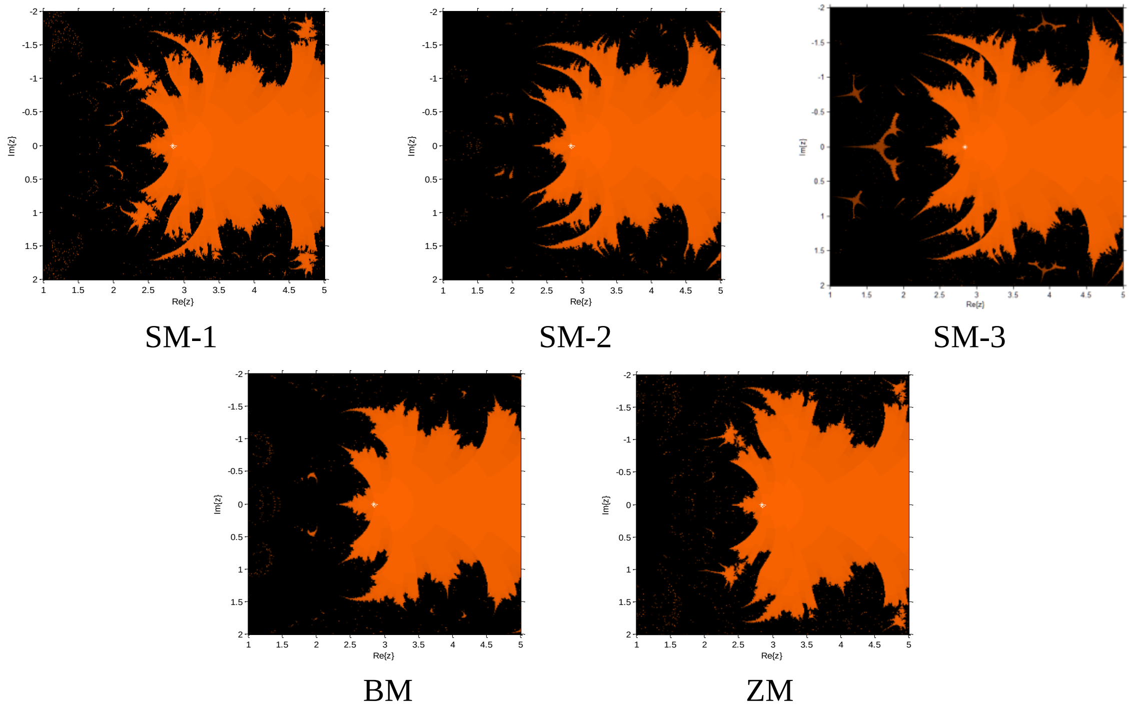


Fig. 2: Basins of attraction of different methods for f_2

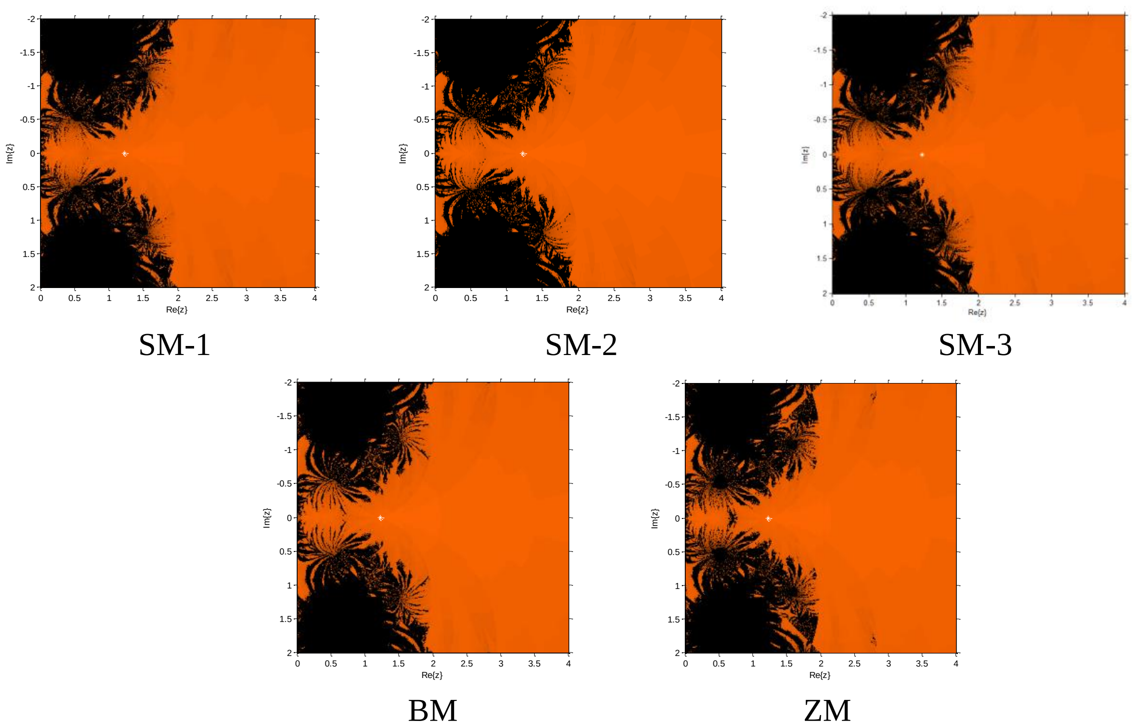
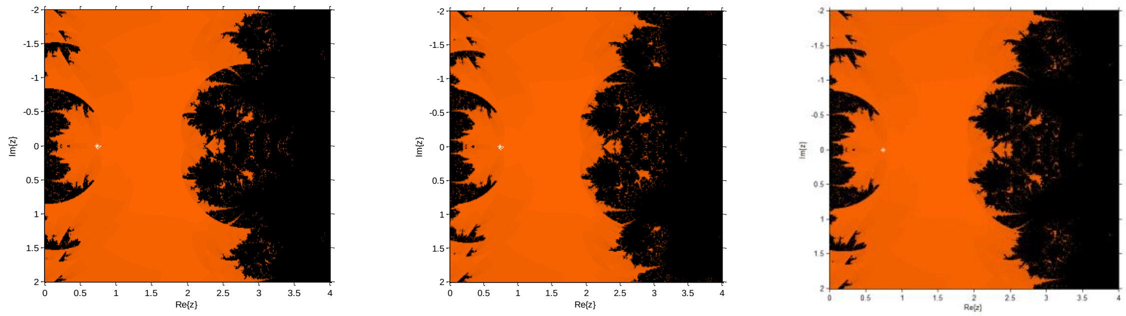
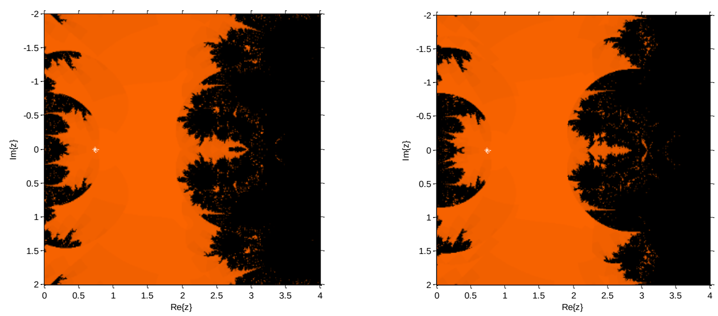


Fig. 3: Basins of attraction of different methods for f_3

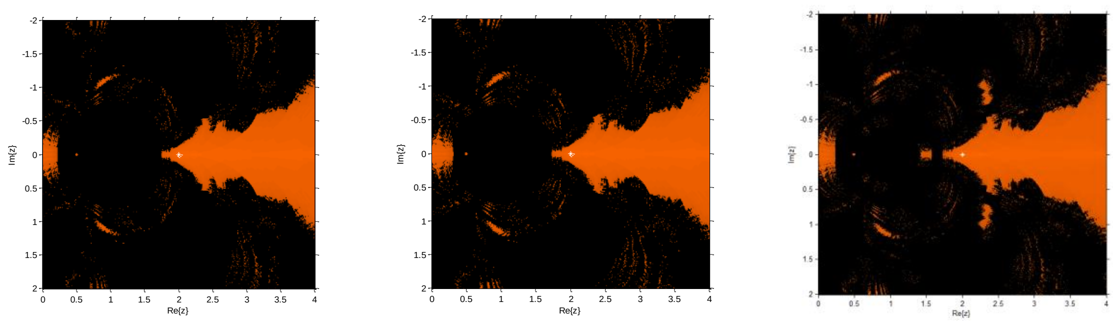


SM-1 SM-2 SM-3

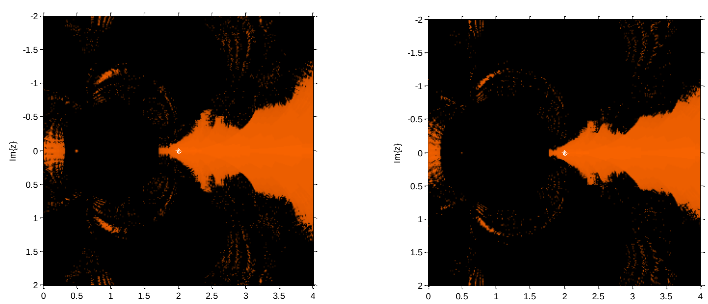


BM ZM

Fig. 4: Basins of attraction of different methods for f_4

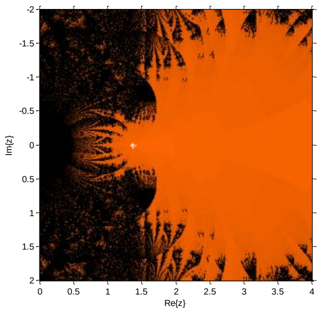


SM-1 SM-2 SM-3

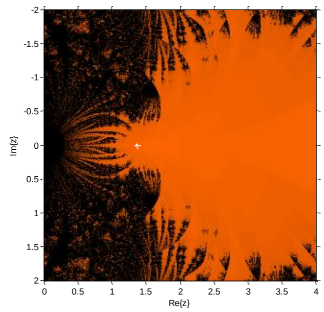


BM ZM

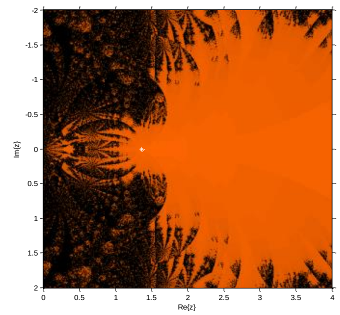
Fig. 5: Basins of attraction of different methods for f_5



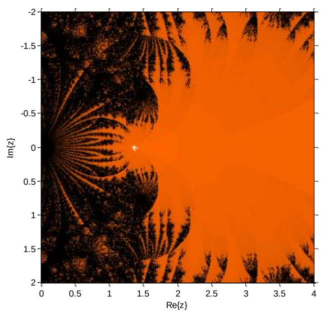
SM-1



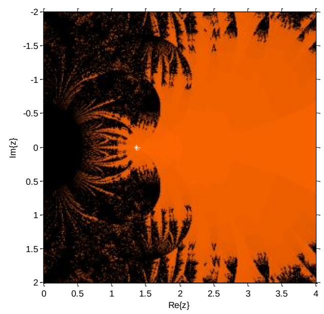
SM-2



SM-3

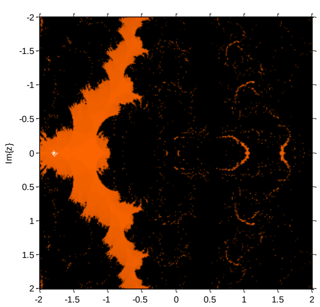


BM

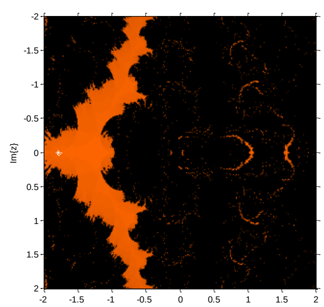


ZM

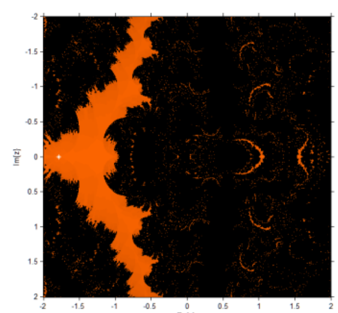
Fig. 6: Basins of attraction of different methods for f_6



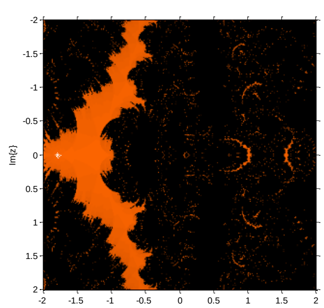
SM-1



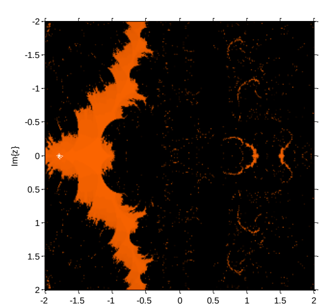
SM-2



SM-3



BM



ZM

Fig. 7: Basins of attraction of different methods for f_7

Conclusion

In this paper, we present a new family of optimal eighth-order methods to find multiple roots of nonlinear equations. An extensive convergence analysis is done which verifies that the new family is optimal eighth order convergent. The proposed family requires four functional evaluations to obtain optimal eighth-order convergence with the efficiency index $8^{\frac{1}{4}} = 1.6817$ which is higher than the efficiency index of any of the methods for multiple roots and of the families of Geum et al. [7, 8]. Finally, numerical and dynamical tests confirm the theoretical results and show that the three members SM-1, SM-2 and SM-3 of the new family are better than existing methods for multiple roots. Hence, the proposed family is efficient among the domain of multiple root finding methods.

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] A. Singh, P. Jaiswal, An efficient family of optimal fourth-order iterative methods for finding multiple roots of nonlinear equations, Proceedings of National Academy of Sciences, India, Section A 85(3), July (2015).
[2] R. Behl, A. Cordero, S.S. Motsa, J.R. Torregrosa, An eighth-order family of optimal multiple root finders and its dynamics. Numer. Algor. (2017). doi:10.1007/s11075-017-0361-6.
[3] R. Behl, A. Cordero, S.S. Motsa, J. R. Torregrosa, V. Kanwar, An optimal fourth-order family of methods for multiple roots and its dynamics, Numer. Algor. 71(4) (2016)775–796.
[4] J. Biazar and B. Ghanbari, A new third-order family of nonlinear solvers for multiple roots, Comput. Math. Appl. 59(10) (2010)3315–3319.
[5] Chicharro, F.I., Cordero, A., Torregrosa, J.R.: Drawing dynamical and pa- rameters planes of iterative families and methods. Sci. World J., 2013, Article ID 780153, 9 pages, (2013)
[6] C. Chun, H. J. Bae, and B. Neta, New families of nonlinear third-order solvers for finding multiple roots, Comput. Math. Appl. 57(9) (2009)1574–1582.
[7] Y.H. Geum, Y.I. Kim, B. Neta, A class of two-point sixth-order multiple-zero finders of modified double-Newton type and their dynamics, Appl. Math. Comput. 270 (2015)387–400.
[8] Y.H. Geum, Y.I. Kim, B. Neta, A sixth-order family of three-point modified Newton-like multiple-root finders and the dynamics behind their extraneous fixed points, Appl. Math. Comput. 283 (2016)120–140.
[9] Jose L. Hueso, Eulalia Martinez, Carles Treuel, Determination of multiple roots of nonlinear equation and applications, J. Math. Chem. 53(3), (2015).
[10] H.T. Kung, J.F. Traub, Optimal order of one-point and multipoint iteration, J. Assoc. Comput. Mach. 21, (1974)643–651.
[11] Siyul Lee, Hyeongmin Choe, On fourth-order iterative methods for multiple roots of nonlinear equation with high efficiency, J.Comput. Anal. Appl. Vol. 18(1),pg 109, Jan (2015).
[12] R.I. Lin, H.M. Ren, Z. Šmarda, Q.B.Wu, Y. Khan and J.L. Hu, New families of third-order iterative methods for finding multiple roots, J. Appl. Math. Vol 2014, Article ID:812072, 9 Pages.
[13] S. Li, L. Cheng, B. Neta, Some fourth-order nonlinear solvers with closed formulae for multiple roots, Comput. Math. Appl. 59(2010)126–135.
[14] Najumudin Ahmad, Vimal Pratap Singh, Some New Three step Iterative methods for solving nonlinear equation using Steffensen's and Halley method, Brit. J. Math. Comp. Sci. 19(2) (2016)1-9. DOI:10.9734/BJMCS/2016/19322.
[15] B. Neta, Extension of Murakami's high-order non-linear solver to multiple roots, Int. J. Comput. Math. 87(5) (2010)1023–1031.
[16] B. Neta, C. Chun, M. Scott, On the development of iterative methods for multiple roots, Appl. Math. Comput. 224 (2013)358–361.
[17] A. M. Ostrowski, Solution of equations and systems of equations, Academic Press, New York (1960).

[18] M.S. Petkovic, B. Neta, L.D. Petkovic, J. Dzunic, Multipoint methods for solving nonlinear equations, Academic Press (2013).
[19] J.R. Sharma, R. Sharma, Modified Jarratt method for computing multiple roots, Appl. Math. Comput. 217 (2010)878–881.
[20] E. Schröder, Über unendlich viele algorithmen zur auflösung der gleichungen,Math. Annal. 2 (1870)317–365.
[21] J.F. Traub, Iterative Methods for the solution of equations. Prentice-Hall, Englewood Cliffs (1964).
[22] F. Zafar, A. Cordero, R. Quratulain, J.R.Torregrosa, Optimal iterative methods for finding multiple roots of nonlinear equations using free parameters, J. Math. Chem. (2017). https://doi.org/10.1007/s10910-017-0813-1.
[23] S. Weerakoon and T. G. I. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett. 13(8) (2000)87–93.
[24] X. Zhou, X. Chen, Y. Song, Constructing higher-order methods for obtaining the multiple roots of nonlinear equations,J. Comput. Appl. Math. 235(2011)4199–4206.