An optimal iteration function for multiple zeros with eighth-order convergence

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Abstract

Over the past several years, many scholars have attempted to construct higher-order schemes for locating multiple solutions of a univariate function having known multiplicity $m \geq 1$. But till date, we have a very limited literature (only four research articles) of eighth-order convergence iteration functions for multiple zeros. The primary contribution of this study is to propose an optimal eighth-order scheme for multiple zeros having simple and compact body structure with faster convergence. An extensive convergence analysis is also present with the main theorem which clearly show the eighth-order convergence of propose iteration scheme. Finally, numerical tests on some real-life problems, such as a Van der Waals equation of state and the conversion problem from the chemical engineering, among others are presented, which confirm the theoretical results to great extent of this study.

Keywords: Nonlinear equations, King-Traub conjecture, multiple roots, optimal iterative methods, efficiency indexce.

Introduction

Finding the multiple zeros of the involved function $f(x) = 0$ (where $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic function in the enclosed region $\mathbb{D}$ containing the required zero) is one of the most challenging, of great significance and difficult tasks in the area of computational mathematics. It is quite tough to obtain exact solution in analytic way of such problems or we can say that it is almost fictitious. So, we have to satisfy ourselves by obtaining approximated and efficient solution up to any specific degree of accuracy by the means of iterative procedure.

This is one of the main reason that researchers are putting their great efforts to resort an iteration function since the past few decades. Additionally, this accuracy is also depend on some other facts like: the considered iterative function, structure of the considered problem, initial guess and programming software namely, Maple, MATLAB, Fortran, Mathematica, etc. Further, the people or researchers using these iterative methods have to struggle with many problems, some of them are like: choice of initial guess/approximation, slower convergence, non-convergence, divergence, oscillation problem close to the initial guess, failure etc. (for the details please see Ostrowski 1960 [17], Traub 1964 [25], Ortega and Rheinboldt 1970 [18], Burden and Faires 2001 [**?**], Petkovic et al. 2012 [19]).

In addition, we dont have a single iteration function which is applicable to every problem until now. This is the main reason that we have an excessive amount of literature on the iteration functions for scalar equations. Here, we concern about the multiple zeros of the involved univariate function in this study. Unfortunately, we have a small amount of literature belongs to higher-order iteration function in the case of scalar equations that can handle multiple roots. The tough calculation work and more time consumption are the main reason behind of this. Moreover, it is more challenging task to construct iterative procedure for multiple zeros as compared to simple.

Eighth-order multi-point methods have faster convergence and better efficiency index as compare to fourth-order [6, 16, 15, 27, 21, 22, 23, 28, 14, 4, 5, 24] and sixth-order [12, 13] iteration functions. Our mean to say that we can save computational time and cost by using them and obtain the approximate solution in a small number of iterations as compared to them. However, we have only four research articles [3, 26, 2, 8] till date that talk about the eighth-order convergence for multiple zeros with known multiplicity $m \geq 1$, according to our best knowledge. But, we know that there is always a scope in the research to obtain better approximation techniques with simple and compact body structure.

While keep all these things in our mind, we not only present an eighth-order iteration scheme having optimal convergence for obtaining the multiple solutions of scalar equation which is better than the existing ones. But, lower residual errors, lower error among two consecutive iterations and more stable computational order of convergence belong to our methods when we compared them to the existing ones of identical order of convergence. Moreover, we present a main theorem which demonstrate the eighth-order convergence when multiplicity of zeros is known in advance. Finally, we give a practical exhibition of our newly propose methods to the real life problems.

Construction of higher-order scheme

In this section, we present the main contribution of this study. Our mean to say that we present an eighth-order scheme for multiple zeros having simple and compact body structure. Therefore, we consider the new scheme in the following way:

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \mu H(\nu)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \kappa\mu\left(G(\mu) + \frac{m\kappa}{1-4\mu}\right)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1}$$

where $\alpha,\ \beta \in \mathbb{R}$ are two free disposable parameters and two weight functions $H : \mathbb{C} \to \mathbb{C}$ and $G : \mathbb{C} \to \mathbb{C}$ are analytic functions in the neighborhoods of (1) and (0) with $\nu = \frac{1+\alpha\mu}{1+\beta\mu},\ \mu = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}},\ \kappa = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$.

In Theorem 1, we illustrate that the constructed scheme (1) attain maximum eighth-order of convergence for all $\alpha, \beta \in \mathbb{R}$ $(\alpha \neq \beta)$, without using any extra functional evaluation. It is interesting to observe that the weight functions H and G play significant role in the construction of scheme (for details please see Theorem 1).

Theorem 1 *Let us consider $x = \xi$ (say) be a multiple zero with multiplicity $m \geq 1$ of the involved function f. In addition, we assume that $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ be an analytic function in the region $\mathbb{D}$ enclosing a multiple zero ξ. Then, the scheme defined by (1) has an eighth-order convergence, when it satisfies the following values*

$$\begin{aligned} &H(1) = m,\ H'(1) = \frac{2m}{\alpha-\beta}\ (\alpha \neq \beta),\ G(0) = m,\ G'(0) = 2m,\ G''(0) = H''(1)(\alpha-\beta)^2 + (2-4\beta)m, \\ &G'''(0) = (\alpha-\beta)^2\Big(H'''(1)(\alpha-\beta) - 6(\beta-1)H''(1)\Big) + 12m(\beta^2 - 2\beta - 2). \end{aligned} \tag{2}$$

Proof: Let us consider that $e_n = x_n - \xi$ and $c_k = \frac{m!}{(m-1+k)!}\frac{f^{m-1+k}(\xi)}{f^m(\xi)},\ k = 2, 3, 4 \ldots, 8$ are the error in nth iteration and asymptotic error constant numbers, respectively. Now, we expand the Taylor's series expansions of the functions $f(x_n)$ and $f'(x_n)$ about $x = \xi$, which are given by

$$f(x_n) = \frac{f^{(m)}(\xi)}{m!} e_n^m \Big(1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)\Big) \quad (3)$$

and

$$\begin{aligned} f'(x_n) =& \frac{f^m(\xi)}{m!} e_n^{m-1} \Big(m + (m+1)c_1 e_n + (m+2)c_2 e_n^2 + (m+3)c_3 e_n^3 + (m+4)c_4 e_n^4 \\ & + (m+5)c_5 e_n^5 + (m+6)c_6 e_n^6 + (m+7)c_7 e_n^7 + (m+8)c_8 e_n^8 + O(e_n^9)\Big), \end{aligned} \quad (4)$$

respectively.
By using the expressions (3) and (4) in the first substep of scheme (1), we have

$$y_n - \xi = \frac{c_1}{m} e_n^2 + \frac{1}{m^2}\Big(2mc_2 - (m+1)c_1^2\Big) e_n^3 + \sum_{i=0}^{4} \theta_i e_n^{i+4} + O(e_n^9), \quad (5)$$

where $\theta_i = \theta_i(m, c_1, c_2, \ldots, c_8)$ are given in terms of $m, c_2, c_3, \ldots, c_8$ for example $\theta_0 = \frac{1}{m^3}\Big[3m^2 c_3 + (m+1)^2 c_1^3 - m(3m+4)c_1 c_2\Big]$ and $\theta_1 = \frac{1}{m^4}\Big[2c_2 c_1^2 m(2m^2 + 5m + 3) - 2c_3 c_1 m^2(2m+3) - 2m^2\Big(c_2^2(m+2) - 2c_4 m\Big) - c_1^4(m+1)^3\Big]$, etc.
With the help of expression (5) and Taylor Series expansion, we further obtain

$$\begin{aligned} f(y_n) =& f^{(m)}(\xi) e_n^{2m} \Bigg[\frac{\left(\frac{c_1}{m}\right)^m}{m!} + \frac{(2mc_2 - (m+1)c_1^2)\left(\frac{c_1}{m}\right)^m e_n}{m! c_1} + \left(\frac{c_1}{m}\right)^{1+m} \frac{1}{2m! c_1^3}\{(3 + 3m + 3m^2 + m^3)c_1^4 \\ & - 2m(2 + 3m + 2m^2)c_1^2 c_2 + 4(m-1)m^2 c_2^2 + 6m^2 c_1 c_3\} e_n^2 + \sum_{i=0}^{4} \bar{\theta}_i e_n^{i+3} + O(e_n^8)\Bigg]. \end{aligned} \quad (6)$$

From the expressions (3) and (6), we have

$$\mu = \frac{c_1 e_n}{m} + \frac{2mc_2 - (m+2)c_1^2}{m^2} e_n^2 + \sum_{i=0}^{4} \bar{\bar{\theta}}_i e_n^{i+3} + O(e_n^8), \quad (7)$$

which further leads us

$$\nu = \frac{\alpha\mu + 1}{\beta\mu + 1} = 1 + (\alpha - \beta) \sum_{k=1}^{8} \gamma_k e_n^k + O(e_n^9), \quad (8)$$

where $\gamma_k = \gamma_k(m, \alpha, \beta, c_1, c_2, \ldots, c_8)$ are given in terms of $m, \alpha, \beta, c_2, c_3, \ldots, c_8$ for example $\gamma_1 = \frac{c_1}{m}$, $\gamma_2 = \frac{1}{m^2}\Big[2c_2 m - c_1^2(\beta + m + 2)\Big]$, $\gamma_3 = \frac{1}{2m^3}\Big[\Big(2\beta^2 + 8\beta + 2m^2 + (4\beta + 7)m + 7\Big)c_1^3 + 6c_3 m^2 - 2c_2 c_1 m(4\beta + 3m + 7)\Big]$, etc.
Now, let us consider $\nu = 1 + \Omega$. Then, from the expression (8) that the remainder $\Omega = \nu - 1$ is infinitesimal with the order e_n. Therefore, we can expand weight function $H(\nu)$ in the neighborhood of (1) by Taylor's series expansion up to third-order terms in the following way:

$$H(\nu) = H(1) + H'(1)\Omega + \frac{1}{2!}H''(1)\Omega^2 + \frac{1}{3!}H'''(1)\Omega^3. \quad (9)$$

By using expressions (3)–(9) in the second substep of scheme (1), we obtain

$$z_n - \xi = -\frac{c_1\Big(H(1) - m\Big)}{m^2} e_n^2 + \sum_{i=0}^{5} A_i e_n^{i+3} + O(e_n^9), \tag{10}$$

where $A_i = A_i(m, c_1, c_2, \ldots, c_8, \alpha, \beta, H(1), H'(1), H''(1), H'''(1))$ are given in terms of $m, c_1, c_2, c_3, \ldots, c_8, \alpha, \beta$, $H(1), H''(1), H'''(1)$. For example, first coefficient explicitly written as $A_0 = \frac{1}{m^3}\Big[2c_2 m\Big(m - H(1)\Big) - c_1^2\Big(m^2 + m - H(1)(m+3) + (\alpha - \beta)H'(1)\Big)\Big]$ and we can also write other ones in the similar way.
It is straightforward to say from the expression (10) that we can easily obtain at least third-order convergence, when we consider

$$H(1) = m. \tag{11}$$

With the help of expression (11) and $A_0 = 0$, we obtain

$$\frac{c_1^2\Big(H'(1)(\beta - \alpha) + 2m\Big)}{m^3} = 0, \tag{12}$$

which further yield

$$H'(1) = \frac{2m}{\alpha - \beta}, \quad \alpha \neq \beta. \tag{13}$$

In this way, we reach optimal fourth-order convergence. Now, by inserting the expressions (11) and (13) in (10), we have

$$z_n - \xi = \left[\frac{\Big(m^2 - H''(1)(\alpha - \beta)^2 + (4\beta + 9)m\Big)c_1^3 - 2c_1 c_2 m^2}{2m^4}\right] e_n^4 + \sum_{i=2}^{5} A_i e_n^{i+3} + O(e_n^9). \tag{14}$$

Again, with the help of Taylor series expansion and expression (14), we obtain

$$f(z_n) = f^{(m)}(\xi) e_n^{4m}\left[\frac{2^{-m}\left(\frac{c_1^3\Big(-H''(1)(\alpha-\beta)^2 + m^2 + (4\beta+9)m\Big) - 2c_1 c_2 m^2}{m^4}\right)^m}{m!} + \sum_{i=1}^{5} \bar{A}_i e_n^i + O(e_n^6)\right]. \tag{15}$$

From the expressions (6) and (15), we further have

$$\kappa = \frac{c_1^2\Big(m^2 - H''(1)(\alpha - \beta)^2 + (4\beta + 9)m\Big) - 2c_2 m^2}{2m^3} e_n^2 + \sum_{i=1}^{5} \bar{\bar{A}}_i e_n^{i+2} + O(e_n^8). \tag{16}$$

It is clear from the expression (16) that the κ is of order e_n^2. Therefore, we can expand weight function $G(\mu)$ in the neighborhood of origin (0) by Taylor's series expansion up to third-order terms in the following way:

$$G(\mu) = G(0) + G'(0)\mu + \frac{1}{2!}G''(0)\mu^2 + \frac{1}{3!}G'''(0)\mu^3. \tag{17}$$

Insert the expressions (3) – (17) in the last substep of scheme (1), we obtain

$$\begin{aligned} e_{n+1} = & \frac{c_1\Big(G(0)-m\Big)\Big[c_1^2\Big(m^2-H''(1)(\alpha-\beta)^2+(4\beta+9)m\Big)-2c_2m^2\Big]}{2m^5}e_n^4 \\ & + \sum_{i=1}^{4} L_i e_n^{i+4} + O(e_n^9), \end{aligned} \tag{18}$$

where $L_i = L_i(\alpha, \beta, m, c_1, c_2, \ldots, c_8, H''(1), H'''(1), G'(0), G''(0), G'''(0))$.
It is noteworthy that we can obtain at least fifth-order convergence if we choose

$$G(0) = m. \tag{19}$$

By using the value of $G(0) = m$ and $L_1 = 0$, we have

$$-\frac{c_1^2\Big(G'(0)-2m\Big)\Big[c_1^2\Big(m^2-H''(1)(\alpha-\beta)^2+(4\beta+9)m\Big)-2c_2m^2\Big]}{2m^6} = 0, \tag{20}$$

which further yield

$$G'(0) = 2m. \tag{21}$$

Again, by inserting the value of $G(0)$ and $G'(0)$ in $L_2 = 0$, we yield

$$-\frac{c_1^3\Big[c_1^2\Big(m^2-H''(1)(\alpha-\beta)^2+(4\beta+9)m\Big)-2c_2m^2\Big]\Big(G''(0)-H''(1)(\alpha-\beta)^2+(4\beta-2)m\Big)}{4m^7} = 0, \tag{22}$$

which further have

$$G''(0) = H''(1)(\alpha-\beta)^2 + (2-4\beta)m. \tag{23}$$

By using the expressions (19), (21) and (23) in $L_3 = 0$, leads us

$$\begin{aligned} & -\frac{c_1^4\Big(c_1^2\Big(-H''(1)(\alpha-\beta)^2+m^2+(4\beta+9)m\Big)-2c_2m^2\Big)}{12m^8} \\ & \times\Big(G'''(0)+(\alpha-\beta)^2(6(\beta-1)H''(1)+H'''(1)(\beta-\alpha))-12m(\beta^2-2\beta-2)\Big) = 0, \end{aligned} \tag{24}$$

which further provide

$$G'''(0) = (\alpha-\beta)^2\Big(H'''(1)(\alpha-\beta)-6(\beta-1)H''(1)\Big)+12m(\beta^2-2\beta-2). \tag{25}$$

In order to obtain final asymptotic error constant term, we insert the expressions (19), (21), (23) and (25) in (18). Then, we have

$$\begin{aligned} e_{n+1} = & \frac{c_1\Big(c_1^2(m^2-H''(1)(\alpha-\beta)^2+(4\beta+9)m)-2c_2m^2\Big)}{24m^9}\Big[c_1^4\Big\{(\alpha-\beta)^2\Big(3(6\beta^2-8\beta+15)H''(1) \\ & -2(3\beta-2)(\alpha-\beta)H'''(1)\Big)-m\Big(24\beta^3-48\beta^2+180\beta+3H''(1)(\alpha-\beta)^2+433\Big)+6(2\beta+1)m^2 \\ & +7m^3\Big\}-6c_2c_1^2m\Big(4m^2-H''(1)(\alpha-\beta)^2+(4\beta+2)m\Big)+12c_3c_1m^3+12c_2^2m^3\Big]e_n^8+O(e_n^9). \end{aligned} \tag{26}$$

The expression (26) demonstrate that our scheme (1) reaches maximum eighth-order convergence for all α and β (provided $\alpha \neq \beta$) by using only four functional evaluations per full iteration. Hence, it is an optimal scheme in the sense of Kung-Traub conjecture, completing the proof. $\square$

Some special cases

In this section, we discuss some special cases of our proposed scheme (1) based on different weight functions $H(\nu)$ and $G(\mu)$. Therefore, we have depicted some special cases of the scheme (1) in Table 1. We can also easily obtain several new eighth-order iterative methods for multiple zeros by choosing different kind of weight functions provided they should satisfy the conditions of Theorem 1.

Table 1: Some special cases of the proposed scheme (1).

Cases	$H(\nu)$	$G(\mu)$
Case-1	$\frac{m(\alpha-\beta+2\nu-2)}{\alpha-\beta}$	$m\Big[1+2\mu+(1-2\beta)\mu^2+2(\beta^2-2\beta-2)\mu^3\Big].$
Case-2	$\frac{m(\alpha-\beta+2\nu-2)}{\alpha-\beta}$	$\frac{m\left(2\beta^2\mu+\beta\left(2-4\mu^2\right)-(3\mu+1)^2\right)}{2\beta^2\mu+\beta(2-4\mu)-4\mu-1}$
Case-3	$a_1+\frac{a_2}{\nu}$ where, $a_1=-\frac{2m}{\alpha-\beta}, a_2=\frac{m(\alpha-\beta+2)}{\alpha-\beta}$	$m\Big[1+2\mu+(1-2\alpha)\mu^2+2(\alpha^2-2\alpha-2)\mu^3\Big]$
Case-4	$a_1+\frac{a_2}{\nu}$ where, $a_1=-\frac{2m}{\alpha-\beta}, a_2=\frac{m(\alpha-\beta+2)}{\alpha-\beta}$	$\frac{m\left(2\alpha^2\mu+\alpha(2-4\mu^2)-(3\mu+1)^2\right)}{2\alpha^2\mu+\alpha(2-4\mu)-4\mu-1}$
Case-5	$\frac{b_1}{\nu}+\frac{b_2}{1+\nu}$ where, $b_1=\frac{m(-\alpha+\beta-4)}{\alpha-\beta}$, $b_2=\frac{4m(\alpha-\beta+2)}{\alpha-\beta}$	$\frac{m}{4}\left(4+8\mu-2b_3\mu^2+b_4\mu^3\right)$ $b_3=\alpha^2-2\alpha(\beta-3)+\beta^2-2\beta-2,$ $b_4=3\alpha^3-5\alpha^2(\beta-2)+\alpha(\beta^2+4\beta-24)+\beta^3-6\beta^2+8\beta-16$

Let us remark that the order of the proposed scheme (1) does not depend on the values of α and β (provided $\alpha \neq \beta$). So, these elements can be considered as free parameters in order to analyze the computational results.

Numerical experiments

In this section, we illustrate the efficiency and convergence behavior of our iteration functions for particular cases. Therefore, we use case-1 for $(\alpha=0,\ \beta=-2)$, $\left(\alpha=\frac{1}{2},\ \beta=-\frac{3}{2}\right)$, $\left(\alpha=\frac{1}{4},\ \beta=-\frac{7}{4}\right)$ and case-2 for $(\alpha=0,\ \beta=-2)$ in expression (1), known by $PM1$, $PM2$, $PM3$ and $PM4$, respectively. In this regards, we choose four real life problems having multiple and simple zeros and two standard academic problems with multiple zeros. The details are outline in the examples (1)–(6).

For better comparison of our iterative methods, we consider several existing methods of order six and eight (optimal). Firstly, we compare our methods with a non optimal family of sixth-order iteration functions given by Geum et al. [13], out of them we choose the case 5YD, which is given by

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\ m \geq 1,\\ w_n &= x_n - m\left[\frac{(u_n-2)(2u_n-1)}{(u_n-1)(5u_n-2)}\right]\frac{f(x_n)}{f'(x_n)},\\ x_{n+1} &= x_n - m\left[\frac{(u_n-2)(2u_n-1)}{(5u_n-2)(u_n+v_n-1)}\right]\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{27}$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$ and $v_n = \left(\frac{f(w_n)}{f(x_n}\right)^{\frac{1}{m}}$, is denoted by GM.

In addition, we demonstrate comparison of them with an optimal eighth-order iteration function proposed by Behl et al. [8], which is given by (this was one of the best scheme claimed by them):

$$\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - mu_n\frac{f'(x_n)}{f'(x_n)}\left[\frac{1+\beta u_n}{(\beta-2)u_n+1}\right],\\
x_{n+1} &= z_n - u_n v_n\frac{f(x_n)}{f'(x_n)}\left[\frac{1}{2}m\Big\{(2v_n+1)\Big(4(\beta^2-6\beta+6)u_n^3+(10-4\beta)u_n^2+4u_n+1\Big)+1\Big\}\right]
\end{aligned} \tag{28}$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$ and $v_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$, is known by BM.

Moreover, we compare them with optimal eighth-order iterative methods constructed by Zafar et al. [26]. We choose the following schemes out of them

$$\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - mu_n\Big(6u_n^3 - u_n^2 + 2u_n + 1\Big)\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= w_n - mu_n v_n(1+2u_n)(1+v_n)\left(\frac{2w_n+1}{A_2P_0}\right)\frac{f(x_n)}{f'(x_n)}
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - mu_n\left(\frac{1-5u_n^2+8u_n^3}{1-2u_n}\right)\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= w_n - mu_n v_n(1+2u_n)(1+v_n)\left(\frac{3w_n+1}{A_2P_0(1+w_n)}\right)\frac{f(x_n)}{f'(x_n)},
\end{aligned} \tag{30}$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $v_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$, $w_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$, with $A_2 = P_0 = 1$ (both schemes (29) and (30)) are known as $FM1$ and $FM2$, respectively.

Finally, we also contrast them with another optimal family of eighth-order methods presented by Behl et al. [2], out of them we choose the following methods

$$\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\
z_n &= x_n - mu_n\left(1+2u_n\right)\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= z_n - \frac{u_n w_n}{1-w_n}\left(\frac{m\left(u_n\left(8v_n+6\right)+9u_n^2+2v_n+1\right)}{4u_n+1}\right)\frac{f(x_n)}{f'(x_n)}
\end{aligned} \tag{31}$$

and

$$
\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)},\\
w_n &= y_n - m u_n\left(1+2u_n\right)\frac{f(x_n)}{f'(x_n)}, \qquad (32)\\
x_{n+1} &= z_n - \frac{u_n w_n}{1-w_n}\left(4u_4^3 - u_4^2 - 2u_4 - 2v_4 - 1\right)\frac{f(x_n)}{f'(x_n)},
\end{aligned}
$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $v_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$, $w_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$, are called by $RM1$ and $RM2$, respectively.

In Tables 2 – 3, we display the number of iteration indexes (n), error in the consecutive iterations $|x_{n+1} - x_n|$, computational order of convergence (ρ) (we used the formula given by Cordero and Torregrosa [10] in order to calculate ρ) and absolute residual error of the corresponding function $(|f(x_n)|)$. We make our calculations with several number of significant digits (minimum 5000 significant digits) to minimize the round off error.

As we mentioned in the above paragraph we calculate the values of all the constants and functional residuals up to several number of significant digits. However, due to the limited paper space, we display the value of errors in the consecutive iterations $|x_{n+1} - x_n|$ and absolute residual errors in the function $|f(x_n)|$ up to 2 significant digits with exponent power which are depicted in Tables 2 – 3. Moreover, computational order of convergence is provided up to 5 significant digits. Finally, we mentioned the values of approximated zeros up to 25 significant digits for each of the examples.

All computations have been performed using the programming package $Mathematica$ 11 with multiple precision arithmetic. Further, the meaning of $a(\pm b)$ is shorthand for $a \times 10^{(\pm b)}$ in the Tables 2–3.

Example 1 Fractional conversion in a chemical reactor:
Let us consider the following expression (for the details of this problem please see [20])

$$
f_1(x) = \frac{x}{1-x} - 5\log\left[\frac{0.4(1-x)}{0.4-0.5x}\right] + 4.45977, \qquad (33)
$$

In the above expression x represents the fractional conversion of species A in a chemical reactor. Since, there will be no physical meaning of above fractional conversion if x is less than zero or greater than one. In this sense, x is bounded in the region $0 \leq x \leq 1$. In addition, our required zero to this problem is $\xi = 0.7573962462537538794596413$. Moreover, it is interesting to note that the above expression will be undefined in the region $0.8 \leq x \leq 1$ which is very close to our desired zero. Furthermore, there are some other properties to this function which make the solution more difficult. The derivative of the above expression will be very close to zero in the region $0 \leq x \leq 0.5$ and there is an infeasible solution for $x = 1.098$.

Example 2 Continuous stirred tank reactor (CSTR)

Let us consider the isothermal continuous stirred tank reactor (CSTR). Components A and R are fed to the reactor at rates of Q and $q - Q$, respectively. Then, we obtain the following reaction scheme in the reactor (for the details see [9]):

$$
\begin{aligned}
A + R &\rightarrow B \\
B + R &\rightarrow C \\
C + R &\rightarrow D \\
C + R &\rightarrow E
\end{aligned}
$$

The problem was analyzed by Douglas [11] in order to design simple feedback control systems. He presented the following expression for the transfer function of the reactor

$$K_C \frac{2.98(x+2.25)}{(x+1.45)(x+2.85)^2(x+4.35)} = -1,$$

where K_C is the gain of the proportional controller. The control system is stable for values of K_C that yields roots of the transfer function having negative real part. If we choose $K_C = 0$ we get the poles of the open-loop transfer function as roots of the nonlinear equation:

$$f_2(x) = x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875. \tag{34}$$

No doubts, the above function f_2 has four zeros $\xi = -1.45, -2.85, -2.85, -4.35$. However, our required zero is $\xi = -4.35$ for expression (34).

Example 3 Van der Waals equation of state

$$\left(P + \frac{a_1 n^2}{V^2}\right)(V - na_2) = nRT,$$

explains the behavior of a real gas by introducing in the ideal gas equations two parameters, α_1 and α_2, specific for each gas. The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V

$$PV^3 - (na_2P + nRT)V^2 + \alpha_1 n^2 V - \alpha_1\alpha_2 n^2 = 0.$$

Given the constants α_1 and α_2 of a particular gas, one can find values for n, P and T, such that this equation has a three simple roots. By using the particular values, we obtain the following nonlinear function

$$f_3(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675.$$

have three zeros and out of them one is a multiple zero $\xi = 1.75$ of multiplicity of order two and other one simple zero $\xi = 1.72$. However, our desired root is $\xi = 1.75$.

Example 4 Multifactor problem

An undesirable RF breakdown which may happen in the high power microwave devices working under the vacuum condition is known as is multifactor [1]. For example, multifactor appears inside a parallel plate waveguide. There exists an electric field with an electric potential difference which creates the electron movement between these two plates. An interesting case in the study of the electron trajectories is when the electron reaches a plate with root of multiplicity 2. The trajectory of an electron in the air gap between two parallel plates is as follows

$$y(t) = y_0 + \left(v_0 + e\frac{E_0}{m\omega}\sin(\omega t_0 + \alpha)\right)(t - t_0) + e\frac{E_0}{m\omega^2}(\cos(\omega t + \alpha) - \cos(\omega t_0 + \alpha)) \tag{35}$$

where m and e are the mass and charge of the electron at rest, $E_0 \sin(\omega t + \alpha)$ is the RF electric field between plates and y_0 and v_0 are the position and velocity of the electron at time t_0. We consider the following particular case of (35), where the parameters have been normalized:

$$f_4(x) = x + \cos(x) - \frac{\pi}{2} \tag{36}$$

with the zero $\xi = \frac{\pi}{2}$ of multiplicity 3.

Example 5 Let us consider a polynomial equation similar to [18], which is given by

$$f_5(x) = ((x-1)^3 - 1)^{100}. \tag{37}$$

The above function has one multiple zero at $\xi = 2$ of multiplicity $m = 100$.

Example 6 Let us consider the following standard nonlinear test function from Behl et al. [6]

$$f_6(x) = \left(1 - \sqrt{1 - x^2} + x + \cos\left(\frac{\pi x}{2}\right)\right)^3. \tag{38}$$

The above function has a multiple zero at $\xi = -0.7285840464448267167123331$ of multiplicity 3.

Conclusion

In this study, we propose a new eighth-order iteration function having optimal eight-order convergence for multiple zeros of a univariate function with faster convergence, simple and compact body structure. The construction of the present scheme is based on the weight function approach that play an important role in the establishment of eighth-order convergence. In addition, we presented an extensive convergence analysis with the main theorem which clearly show the eighth-order convergence. Each member of our scheme is optimal in the sense of the classical Kung-Traub conjecture. The computational efficiency index is defined as $E = p^{1/\theta}$, where p is the order of convergence and θ is the number of functional evaluations per iteration. Thus, the efficiency index of the present methods is $E = \sqrt[4]{8} \approx 1.682$ which is better than the classical Newton's method $E = \sqrt[2]{2} \approx 1.414$.

Moreover, we can easily obtain several new methods by considering different weight functions in our scheme (1). Lower residual errors, lower error among two consecutive iteration and stable computational order of convergence belongs to our methods when we compared them to the existing ones of same order on problem like chemical conversion, continuous stirred tank reactor, Van der Waals equation of state, multi factor problem, etc. Finally, on accounts of the results obtained, it can be concluded that our proposed methods are highly efficient and perform better than the existing methods.

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Table 2: Difference between two consecutive iterations ($|x_{n+1} - x_n|$) of different iteration functions.

$f(x)$	n	GM	BM	$FM1$	$FM2$	$RM1$	$RM2$	$PM1$	$PM2$	$PM3$	$PM4$
$f_1(x)$	1	$1.8(-10)$	$5.1(-12)$	$5.1(-11)$	$7.7(-11)$	$8.0(-12)$	$1.4(-11)$	$8.2(-13)$	$9.4(-13)$	$2.9(-12)$	$1.3(-14)$
	2	$1.7(-53)$	$1.2(-81)$	$1.6(-72)$	$5.9(-71)$	$1.4(-79)$	$9.4(-78)$	$8.1(-89)$	$5.8(-88)$	$9.7(-84)$	$4.3(-105)$
	3	$1.3(-311)$	$1.5(-638)$	$1.5(-564)$	$7.3(-552)$	$1.2(-621)$	$4.7(-607)$	$7.3(-697)$	$1.3(-689)$	$1.4(-655)$	$7.4(-829)$
	ρ	6.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
$f_2(x)$	1	$9.5(-3)$	$2.0(-2)$	$2.0(-2)$	$2.0(-2)$	$2.7(-4)$	$2.7(-4)$	$2.0(-2)$	$2.0(-2)$	$2.0(-2)$	$2.0(-2)$
	2	$8.1(-16)$	$4.2(-18)$	$5.2(-18)$	$5.2(-18)$	$9.1(-14)$	$9.1(-14)$	$4.2(-18)$	$4.2(-18)$	$4.2(-18)$	$4.2(-18)$
	3	$3.9(-94)$	$3.1(-143)$	$1.9(-142)$	$1.7(-142)$	$3.4(-42)$	$3.4(-42)$	$3.0(-143)$	$3.0(-143)$	$3.0(-143)$	$3.0(-143)$
	ρ	5.9929	7.9858	7.9846	7.9847	3.0005	3.0005	7.9862	7.9861	7.9861	7.9862
$f_3(x)$	1	$3.9(-4)$	$2.6(-4)$	$3.9(-4)$	$4.1(-4)$	$2.6(-4)$	$2.7(-4)$	$7.2(-5)$	$2.9(-5)$	$5.1(-5)$	$3.3(-5)$
	2	$1.0(-14)$	$3.6(-19)$	$5.2(-17)$	$9.8(-17)$	$1.4(-19)$	$1.1(-18)$	$9.4(-24)$	$1.1(-27)$	$3.9(-25)$	$2.4(-27)$
	3	$3.9(-78)$	$6.1(-138)$	$5.9(-120)$	$1.2(-117)$	$1.0(-141)$	$6.1(-134)$	$8.0(-175)$	$3.3(-207)$	$5.0(-186)$	$7.5(-207)$
	ρ	5.9975	7.9977	7.9945	7.9941	8.0026	7.9971	7.9994	7.9996	7.9995	7.9995
$f_4(x)$	1	$2.5(-6)$	$4.3(-6)$	$4.3(-6)$	$4.3(-6)$	$1.4(-10)$	$1.4(-10)$	$1.3(-17)$	$1.3(-17)$	$1.3(-17)$	$1.3(-17)$
	2	$1.5(-18)$	$1.4(-30)$	$1.4(-30)$	$1.4(-30)$	$3.8(-52)$	$3.8(-52)$	$1.4(-30)$	$1.4(-30)$	$1.4(-30)$	$1.4(-30)$
	3	$3.7(-55)$	$5.9(-153)$	$5.9(-153)$	$5.9(-153)$	$5.3(-260)$	$5.3(-260)$	$5.9(-153)$	$5.9(-153)$	$5.9(-153)$	$5.9(-153)$
	ρ	3.0000	5.0000	5.0000	5.000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
$f_5(x)$	1	$2.0(-7)$	$9.5(-8)$	$4.8(-7)$	$6.5(-7)$	$6.3(-8)$	$1.9(-7)$	$5.1(-8)$	$2.3(-8)$	$2.7(-8)$	$1.5(-8)$
	2	$1.8(-41)$	$1.6(-55)$	$5.7(-49)$	$8.4(-48)$	$4.2(-57)$	$8.0(-53)$	$8.0(-58)$	$2.6(-59)$	$4.6(-60)$	$1.7(-15)$
	3	$1.0(-245)$	$1.3(-437)$	$2.2(-384)$	$6.6(-375)$	$5.9(-169)$	$9.6(-416)$	$2.9(-456)$	$3.2(-454)$	$3.2(-474)$	$1.9(-118)$
	ρ	6.0000	8.0000	8.0000	8.0000	2.2745	8.0000	8.0000	8.0000	8.0000	14.862
$f_6(x)$	1	$3.5(-6)$	$1.7(-7)$	$2.4(-7)$	$2.4(-7)$	$9.3(-8)$	$9.7(-8)$	$1.2(-7)$	$1.2(-7)$	$1.2(-7)$	$1.1(-7)$
	2	$1.2(-32)$	$4.4(-53)$	$2.0(-51)$	$2.5(-51)$	$3.0(-55)$	$5.8(-55)$	$1.2(-54)$	$1.2(-54)$	$1.2(-54)$	$2.6(-55)$
	3	$1.8(-191)$	$9.4(-418)$	$5.3(-404)$	$3.6(-403)$	$3.1(-435)$	$1.0(-432)$	$8.7(-431)$	$8.7(-431)$	$1.3(-430)$	$2.8(-436)$
	ρ	6.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000

($*$ means the corresponding fails to work. $**$ means computational order of convergence is not calculated in the case divergence.)

Table 3: Comparison based on residual error (i.e. $|f(x_n)|$) of different iteration functions.

$f(x)$	n	GM	BM	$FM1$	$FM2$	$RM1$	$RM2$	$PM1$	$PM2$	$PM3$	$PM4$
	1	$1.4(-8)$	$4.1(-10)$	$4.1(-9)$	$6.1(-9)$	$6.4(-10)$	$1.1(-9)$	$6.6(-11)$	$7.5(-11)$	$2.3(-10)$	$1.0(-12)$
$f_1(x)$	2	$1.4(-51)$	$9.9(-80)$	$1.3(-70)$	$4.7(-69)$	$1.1(-77)$	$7.5(-76)$	$6.5(-87)$	$4.7(-86)$	$7.8(-82)$	$3.4(-103)$
	3	$1.0(-309)$	$1.2(-636)$	$1.2(-562)$	$5.8(-550)$	$9.7(-620)$	$3.8(-605)$	$5.8(-695)$	$1.0(-687)$	$1.1(-653)$	$5.9(-827)$
	1	$1.9(-4)$	$8.0(-4)$	$8.5(-4)$	$8.5(-4)$	$1.5(-7)$	$1.5(-7)$	$8.0(-4)$	$8.0(-4)$	$8.0(-4)$	$8.0(-4)$
$f_2(x)$	2	$1.4(-30)$	$3.7(-35)$	$5.7(-35)$	$5.6(-35)$	$1.7(-26)$	$1.7(-26)$	$3.7(-35)$	$3.7(-35)$	$3.7(-35)$	$3.7(-35)$
	3	$3.2(-187)$	$2.0(-285)$	$7.3(-284)$	$6.3(-284)$	$2.5(-83)$	$2.5(-83)$	$1.9(-285)$	$1.9(-285)$	$1.9(-285)$	$1.9(-285)$
	1	$4.6(-9)$	$2.0(-9)$	$4.6(-9)$	$5.1(-9)$	$2.0(-9)$	$2.3(-9)$	$1.6(-10)$	$2.5(-11)$	$7.7(-11)$	$3.2(-11)$
$f_3(x)$	2	$3.2(-30)$	$4.0(-39)$	$8.0(-35)$	$2.9(-34)$	$5.9(-40)$	$3.4(-38)$	$2.6(-48)$	$3.3(-56)$	$4.6(-51)$	$1.7(-55)$
	3	$4.6(-157)$	$1.1(-276)$	$1.1(-240)$	$4.3(-236)$	$3.1(-284)$	$1.2(-268)$	$1.9(-350)$	$3.3(-415)$	$7.6(-373)$	$1.7(-410)$
	1	$2.6(-18)$	$1.3(-17)$	$1.3(-17)$	$1.3(-17)$	$4.7(-31)$	$4.7(-31)$	$4.7(-31)$	$4.7(-31)$	$4.7(-31)$	$4.7(-31)$
$f_4(x)$	2	$6.2(-55)$	$5.0(-91)$	$5.0(-91)$	$5.0(-91)$	$9.1(-156)$	$9.1(-156)$	$5.0(-91)$	$5.0(-91)$	$5.0(-91)$	$5.0(-91)$
	3	$8.4(-165)$	$3.5(-458)$	$3.5(-458)$	$3.5(-458)$	$2.4(-779)$	$2.4(-779)$	$3.5(-458)$	$3.5(-458)$	$3.5(-458)$	$3.5(-458)$
	1	$1.1(-622)$	$2.2(-655)$	$4.4(-585)$	$5.3(-572)$	$3.9(-673)$	$3.1(-626)$	$4.5(-682)$	$1.3(-709)$	$2.5(-709)$	$3.7(-736)$
$f_5(x)$	2	$9.7(-4027)$	$1.4(-5431)$	$1.2(-4777)$	$8.7(-4661)$	$11.(-5590)$	$9.1(-5163)$	$1.6(-5662)$	$6.7(-5376)$	$2.3(-5886)$	$5.3(-1429)$
	3	$5.4(-24451)$	$4.1(-43641)$	$2.7(-38318)$	$5.1(-37371)$	$1.1(-16775)$	$7.8(-41455)$	$3.3(-45506)$	$6.1(-41287)$	$1.6(-47302)$	$5.9(-11726)$
	1	$1.1(-6)$	$1.3(-20)$	$3.5(-20)$	$3.7(-20)$	$2.1(-21)$	$2.3(-21)$	$4.8(-21)$	$4.8(-21)$	$4.5(-21)$	$3.5(-21)$
$f_6(x)$	2	$4.3(-96)$	$2.2(-157)$	$2.1(-152)$	$4.2(-152)$	$6.7(-164)$	$5.1(-163)$	$4.3(-162)$	$4.3(-162)$	$4.6(-162)$	$4.7(-164)$
	3	$1.5(-572)$	$2.1(-1251)$	$3.8(-1210)$	$1.2(-1207)$	$7.5(-1304)$	$2.5(-1296)$	$1.7(-1290)$	$1.7(-1290)$	$6.2(-1290)$	$5.4(-1307)$

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