Ball convergence for a multi-step Harmonic mean Newton-like method in Banach space

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Abstract

In this paper, we present a local convergence analysis of some iterative methods to approximate a locally unique solution of nonlinear equation in a Banach space setting. In the earlier study, Babajee et al. [1] demonstrate convergence of their methods under hypotheses on the fourth-order derivative or even higher. However, only first-order derivative of the function appears in their proposed scheme. In this study, we have shown that the local convergence of these methods depends under hypotheses only on the first-order derivative and the Lipschitz condition. In this way, we not only expand the applicability of these methods but also proposed the theoretical radius of convergence of these methods. Finally, a variety of concrete numerical examples demonstrate that our results even apply to solve those nonlinear equations where earlier studies cannot apply.

Keywords: Newton-like method, local convergence, Banach space, Lipschitz constant, radius of convergence.

Introduction

One of the most basic and important problem of Numerical analysis concerns with approximating a locally unique solution x^* of the equation of the form

$$F(x) = 0, \tag{1}$$

where F is a Fréchet -differentiable operator defined on a convex subset $\mathbb{D}$ of a Banach space $\mathbb{X}$ with value in a Banach space $\mathbb{Y}$.

Analytical methods for such type of problems are very rare or almost non existent. Therefore, it is only possible to approximate solutions by relying iterative methods. The convergence analysis of iterative methods is usually divided into two categories: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. A very important problem in the study of iterative procedures is the convergence domain. Therefore, it is very important to propose the radius of convergence of the iterative methods.

We study the local convergence of the two step method defined for each $n = 0,\ 1\ 2,\ \ldots$ by

$$\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n),\\
x_{n+1} &= G_{4th\ HM}(x_n) = x_n - H_1(x_n)A(x_n)F(x_n),\\
z(x_n) &= F'(x_n)^{-1}F'(y_n),\\
H_1(x_n) &= I - \frac{1}{4}\Big(z(x_n) - I\Big) + \frac{1}{2}\Big(z(x_n) - I\Big)^2,\\
A(x_n) &= \frac{1}{2}\Big(F'(x_n)^{-1} + F'(y_n)^{-1}\Big)
\end{aligned} \tag{2}$$

and

$$\begin{aligned} x_{n+1} &= G_{(2s+4)th\ HM}(x_n) = z_s(x_n) \\ z_j(x_n) &= z_{j-1}(x_n) - H_2(x_n)A(x_n)F\Big(z_{j-1}(x_n)\Big),\ j = 1,\ 2,\ \ldots,\ s,\ s \geq 1, \\ H_2(x_n) &= 2I - z(x_n), \\ z_0(x_n) &= G_{4th\ HM}(x_n), \end{aligned} \tag{3}$$

where, s is a natural number with $s = 0$, $x_0 \in D$ is an initial point and I is the identity operator. Notice that if $s = 0$, then method (3) reduces to method (2). These methods were studied in [1] in the special case when $\mathbb{X} = \mathbb{Y} = \mathbb{R}^i$ (i is a natural integer). Method (2) was shown to be of order four and method (3) was to be shown of order $2s + 4$. However, the local convergence was shown in [1], by using the Taylor series expansions and hypotheses reaching up to the fifth Fréchet derivative of involved operator F although only first order derivative appears in the proposed schemes. The hypotheses on the derivatives of F restrict the applicability of method (2) and method (3). As a motivational example, define function F on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $D = [-\frac{5}{2}, \frac{1}{2}]$ by

$$F(x) = \begin{cases} x^3 lnx^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then, we have that

$$F'(x) = 3x^2 lnx^2 + 5x^4 - 4x^3 + 2x^2,$$

$$F''(x) = 6xlnx^2 + 20x^3 - 12x^2 + 10x$$

and

$$F'''(x) = 6lnx^2 + 60x^2 - 24x + 22.$$

Then, obviously, function $F'''(x)$ is unbounded on $\mathbb{D}$ at the point $x = 0$. Hence, the results in [1], cannot apply to show the convergence of method (2) or its special cases requiring hypotheses on the fourth derivative of function F or higher. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations [2, 3, 4, 5, 6, 1, 7, 8, 9, 10, 11, 12, 13, 14, 15]. These results show that initial guess should be close to the required root for the convergence of the corresponding methods. But, how close initial guess should be required for the convergence of the corresponding method? These local results give no information on the radius of the ball convergence for the corresponding method. The same technique can be used to other methods.

In the present study we expand the applicability of method (2) using only hypotheses on the first order derivative of function F. We also proposed the computable radii of convergence and error bounds based on the Lipschitz constants. We further present the range of initial guess x^* that tell us how close the initial guess should be required for granted convergence of the method (2). This problem was not addressed in [1].

Local convergence in Banach space

We present the local convergence analysis that follows is based on some scalar functions and parameters. Let $L_0,\ L > 0$ and $M \in [1,\ 3)$ be given parameters. Define functions $g_1,\ g_2,\ h_2,\ p,$

q_j, h_{q_j}, $j = 1, 2, \ldots, s$ (s is a natural integer) on the interval $\left[0,\ \frac{1}{L_0}\right)$ by

$$
\begin{aligned}
g_1(t) &= \frac{1}{2(1-L_0t)}\left(Lt + \frac{2M}{3}\right),\\
g_2(t) &= \frac{Lt}{2(1-L_0t)} + \frac{L_0M(1+g_1(t))t^2}{2(1-L_0t)^2} + \left[\frac{L_0(1+g_1(t))t^2}{4(1-L_0t)^2} + \frac{1}{2}\frac{L_0^2(1+g_1(t))^2t^4}{(1-L_0t)^4}\right]\frac{M^2}{(1-L_0t)^2},\\
h_2(t) &= g_2(t) - 1,\\
p(t) &= \frac{M}{1-L_0t}\left(1 + \frac{L_0(1+g_1(t))}{1-L_0t}\right),\\
q_1(t) &= (1+p(t))g_2(t),\\
h_{q_1}(t) &= q_1(t) - 1,\\
q_j(t) &= (1+p(t))q_{j-1}(t) = (1+p(t))^j g_2(t),\ j = 2,\ 3,\ \ldots,\ s,\\
h_{q_j}(t) &= q_j(t) - 1
\end{aligned}
$$

and parameters r_1 and r_A by

$$
\begin{aligned}
r_1 &= \frac{2\left(1-\frac{M}{3}\right)}{2L_0+L},\\
r_A &= \frac{2}{2L_0+L}.
\end{aligned}
$$

We have that

$$0 < r_1 < r_A, \tag{4}$$

$g_1(r_1) = 1$ and for each $t \in [0,\ r_1] : 0 \le g_1(t) < 1$. Moreover, by the definition of the functions g_2 and $h_2 : h_2(0) = -1$ and $h_2(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. It then follows from the intermediate value theorem that the function h_2 has zeros in the interval $\left(0,\ \frac{1}{L_0}\right)$. Further, consider that r_2 is the smallest such zero. Similarly, we have that $h_{q_j}(0) = -1$ and $h_{q_j}(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. Denote by r_{q_j} the smallest zeros of the functions h_{q_j}, respectively on the interval $\left(0,\ \frac{1}{L_0}\right)$. In particular, we have $h_{q_1}(r_2) = (1+p(r_2))g_2(r_2) - 1 = p(r_2) > 0$, since $1 - L_0r_2 > 0$ and $g_2(r_2) = 1$. Hence, $r_{q_1} < r_2$. Similarly, we get $h_{q_j}(r_{j-1}) = p(r_{q_{j-1}}) > 0$, since $1 - L_0r_{q_{j-1}} > 0$ and $q_{j-1}(r_{q_{j-1}}) = 1$. That is we obtain that

$$r_{q_s} < r_{q_{s-1}} < \cdots < r_{q_1} < r_2. \tag{5}$$

Define

$$r = \min\{r_1,\ r_{q_s}\}. \tag{6}$$

Then, in view of (4) – (6), we have that

$$0 < r < r_A < \frac{1}{L_0} \tag{7}$$

and for each $t \in [0,\ r)$

$$0 \le g_1(t) < 1, \tag{8}$$

$$0 \le g_2(t) < 1, \tag{9}$$

and

$$0 \le q_j(t) < 1. \tag{10}$$

Notice that if $s = 0$, then the radius of convergence should be defined by $r = \min\{r_1,\ r_2\}$. Let $U(w,\ \rho)$, $\bar{U}(w,\ \rho)$ stand, respectively for the open and closed balls in X with center $w \in X$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (3) using the preceding notations.

Theorem 1 *Let $F : D \subseteq \mathbb{X} \to \mathbb{Y}$ be a Fréchet differentiable operator. Suppose there exist $x^* \in D$ and $L_0 > 0$ such that for each $x \in D$*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(\mathbb{Y},\ \mathbb{X}) \tag{11}$$

and

$$\left\| F'(x^*)^{-1}(F'(x) - F'(x^*) \right\| \le L_0 \|x - x^*\|. \tag{12}$$

Moreover, suppose that there exist $L > 0$ and $M \in [1,\ 3)$ such that for each $x,\ y \in D \cap U\left(x^*,\ \frac{1}{L_0}\right)$ the following estimates hold

$$\left\| F'(x^*)^{-1} \left(F'(x) - F'(y)\right) \right\| \le L\|x - y\|, \tag{13}$$

$$\left\| F'(x^*)^{-1} F'(x) \right\| \le M, \tag{14}$$

and

$$\bar{U}\left(x^*,\ r\right) \subseteq D, \tag{15}$$

where the radius of convergence r is defined by (6)*. Then, the sequence $\{x_n\}$ generated by method* (3) *for $x_0 \in U(x^*,\ r) - \{x^*\}$ is well defined, remains in $U(x^*,\ r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold*

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{16}$$

$$\|x_{n+1} - x^*\| = \|z_0(x_n) - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{17}$$

and

$$\begin{aligned} \|z_j(x_n) - x^*\| &\le \Big(1 + p(\|x_n - x^*\|)\Big)\|z_{j-1}(x_n) - x^*\| \\ &\le \Big(1 + p(r)\Big)^j g_2(r)\|x_n - x^*\| < \|x_n - x^*\|, \end{aligned} \tag{18}$$

where the "g" functions are defined previously. Furthermore, for $T \in \left[r,\ \frac{2}{L_0}\right)$, the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*,\ T) \cap D$.

Proof: Mathematical induction shall be used to show estimates (16) – (18). By hypotheses $x_0 \in U(x^*,\ r) - \{x^*\}$, (7) and (12), we get that

$$\left\| F'(x^*)^{-1} \left(F'(x_0) - F'(x^*)\right) \right\| \le L_0\|x - x^*\| < L_0 r < 1. \tag{19}$$

In view of (19) and the Banach Lemma on invertible operators [5, 13], we get that $F'(x_0)^{-1} \in L(Y,\ X),\ y_0$ exists and

$$\left\| F'(x_0)^{-1} F'(x^*) \right\| \le \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{20}$$

We also get that y_0 is well defined by the first sub step of method (3) for $n = 0$. We can write by (11)

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta \tag{21}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*,\ r)$. Then, by (14) and (21), we have that

$$\left\|F'(x^*)^{-1}F'(x_0)\right\| \leq M\|x_0 - x^*\| \tag{22}$$

We can write by the first sub step of method (3) and (11)

$$\begin{aligned} y_0 - x^* = &\left(F'(x_0)^{-1}F'(x^*)\right) \int_0^1 F'(x^*)^{-1} \left(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\right) (x_0 - x^*)d\theta \\ &+ \frac{1}{3}\left(F'(x_0)^{-1}F'(x^*)\right)\left(F'(x^*)^{-1}F'(x_0)\right) \end{aligned} \tag{23}$$

Using (7), (8), (13), (20), (22) and (23), we obtain in turn that

$$\begin{aligned} \|y_0 - x^*\| = &\left\|F'(x_0)^{-1}F'(x^*)\right\| \left\|\int_0^1 F'(x^*)^{-1} \left(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\right)(x_0 - x^*)d\theta\right\| \\ &+ \frac{1}{3}\left\|F'(x_0)^{-1}F'(x^*)\right\| \left\|F'(x^*)^{-1}F'(x_0)\right\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L\|x_0 - x^*\|)} + \frac{M\|x_0 - x^*\|}{3(1 - L\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{24}$$

which shows (16) for $n = 0$ and $y_0 \in U(x^*,\ T)$. It follows from (19), (20) and (24) for y_0 replacing x_0, since $y_0 \in U(x^*,\ T)$ that $F'(y_0)^{-1} \in L(\mathbb{Y},\ \mathbb{X})$

$$\begin{aligned} \left\|F'(y_0)^{-1}F'(x^*)\right\| &\leq \frac{1}{1 - L_0\|y_0 - x^*\|} \\ &\leq \frac{1}{1 - L_0 g_1(\|x_0 - x^*\|)} \\ &\leq \frac{1}{1 - L_0\|x_0 - x^*\|} \end{aligned} \tag{25}$$

and x_1 is well defined by the second sub step of the method (3) for $n = 0$. Then, we can write by the second sub step of method (3) for $n = 0$ in turn that but using (9) instead of (8), we obtain in turn that

$$
\begin{aligned}
x_1 - x^* &= (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + \frac{1}{2}\Big(F'(x_0)^{-1}F(x^*)\Big)\Big[F'(x^*)^{-1}\Big(F'(x_0) - F'(x^*)\Big) \\
&\quad + F'(x^*)^{-1}\Big(F'(x^*) - F'(y_0)\Big)\Big]\Big(F'(y_0)^{-1}F(x^*)\Big)\Big(F'(x^*)^{-1}F(x_0)\Big) \\
&\quad + \Bigg[\frac{1}{4}\Big(F'(x_0)^{-1}F(x^*)\Big)\Big(F'(x^*)^{-1}(F'(y_0) - F'(x^*)) + F'(x^*)^{-1}(F'(x^*) - F'(x_0))\Big) \\
&\quad \times \Big(F'(y_0)^{-1}F'(x^*)\Big)F'(x^*)^{-1} + \frac{1}{2}\Bigg(\Big(F'(x_0)^{-1}F(x^*)\Big)\Big(F'(x^*)^{-1}(F'(y_0) - F'(x^*)) \\
&\quad + F'(x^*)^{-1}(F'(x^*) - F'(x_0))\Big)\Big(F'(y_0)^{-1}F'(x^*)\Big)F'(x^*)\Bigg)^2\Bigg]\frac{1}{2}\Big(F'(x_0)^{-1}F'(x^*)^{-1}\Big) \\
&\quad \times \Big(F'(x^*)^{-1}F'(y_0) + F'(x^*)^{-1}F'(x_0)\Big)\Big(F'(y_0)^{-1}F(x^*)\Big)\Big(F'(x^*)^{-1}F(x_0)\Big).
\end{aligned} \tag{26}
$$

Then, using the triangle inequality in (26), (7), (9), (20), (22) (for $x_0 = x_0$ and $x_0 = y_0$), (24) and (25), we get in turn that

$$
\begin{aligned}
\|x_1 - x^*\| &\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0(\|x_0 - x^*\|))} + \frac{L_0 M\Big(\|x_0 - x^*\| + \|y_0 - x^*\|\Big)\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)(1 - L_0\|y_0 - x^*\|)} \\
&\quad + \Bigg[\frac{L_0\Big(\|x_0 - x^*\| + \|y_0 - x^*\|\Big)\|x_0 - x^*\|}{4(1 - L_0\|x_0 - x^*\|)(1 - L_0\|y_0 - x^*\|)} \\
&\quad + \frac{1}{2}\left(\frac{L_0\Big(\|x_0 - x^*\| + \|y_0 - x^*\|\Big)\|x_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)(1 - L_0\|y_0 - x^*\|)}\right)^2\Bigg]\frac{M^2\|x_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)(1 - L_0\|y_0 - x^*\|)} \\
&\quad \le g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned} \tag{27}
$$

which shows (17) for $n = 0$ and $x_1 \in U(x^*,\ T)$. Notice that by the definition of the method (3), $x_1 = z_0(x_0)$ and all iterates $z_j(x_0),\ j = 1,\ 2,\ \cdots, s$ are well defined. Then, we have by the third sub step of method (3) for $n = 0$

$$
\begin{aligned}
\|z_1(x_0) - x^*\| &\le \|z_0(x_0) - x^*\| + \|H_2(x_0)A(x_0)F'(x^*)\|M\|z_0(x_0) - x^*\| \\
&= (1 + M\|H_2(x_0)A(x_0)F'(x^*)\|)\|z_0(x_0) - x^*\|.
\end{aligned} \tag{28}
$$

We need an upper bound on the norm inside (28). It follows from the definition of $A,\ H_2$ and (27) that

$$
\begin{aligned}
\|H_2(x_0)A(x_0)F'(x^*)\| &\le \frac{1}{2}\left(\frac{1}{1 - L_0\|x_0 - x^*\|} + \frac{1}{1 - L_0\|y_0 - x^*\|}\right)(\|I\| + \|I - u(x_0)\|) \\
&\le \frac{1}{1 - L_0\|x_0 - x^*\|}\Big(1 + \Big\|F'(x_n)^{-1}(F'(x_n) - F'(y_n))\Big\|\Big) \\
&\le \frac{1}{1 - L_0\|x_0 - x^*\|}\left(1 + \frac{L_0(\|x_0 - x^*\| + \|y_0 - x^*\|)}{1 - L_0\|x_0 - x^*\|}\right) \\
&\le \frac{1}{1 - L_0\|x_0 - x^*\|}\left(1 + \frac{L_0(1 + g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - L_0\|x_0 - x^*\|}\right) \\
&\le \frac{p(\|x_0 - x^*\|)}{M}.
\end{aligned} \tag{29}
$$

Using (28) and (29), we get that

$$\begin{aligned} \|z_1(x_0) - x^*\| &\leq \Big(1 + p(\|x_0 - x^*\|)\Big)\|z_0(x_0) - x^*\| \\ &\leq \Big(1 + p(\|x_0 - x^*\|)\Big)g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &= q_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{30}$$

so $z_1(x_0) \in U(x^*,\ r)$ and (18) for $n = 0$ and $j = 1$. In an analogous way by using $z_j(x_0),\ z_{j-1}(x_0)$ instead of $z_1(x_0),\ z_0(x_0)$ in (28) and (29), we get that (18) holds for $n = 0$ and $j = 1,\ 2,\ \ldots,\ s$. Hence, x_2 is well defined in $x_2 \in U(x^*,\ r)$ and by (18) and the definition of the method (3) $\|x_2 - x_1\| \leq \|x_1 - x^*\|$. Continuing in this way, we arrive at the estimates (16) –(18) and $\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r,\ c = g_2(\|x_0 - x^*\|) \in [0,\ 1)$, which shows $x_{k+1} \in U(x^*,\ r)$ and $\lim_{k\to\infty} x_k = x^*$. Finally, to show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ be such that $F(y^*) = 0$. Set $Q = \int_0^1 F'\left(x^* + \theta(y^* - x^*)\right) d\theta$. Then, using (12), we get that

$$\left\|F'(x^*)^{-1}(Q - F'(x^*))\right\| \leq L_0 \int_0^1 \theta\|x^* - y^*\|d\theta = \frac{L_0}{2}T < 1. \tag{31}$$

Hence, $Q^{-1} \in L(Y,\ X)$. Then, in view of the identity $F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. □

Remark

(a) In view of (12) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*) + F'(x^*))\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\ &\leq 1 + L_0\|x_0 - x^*\|, \end{aligned}$$

condition (14) can be dropped and M can be replaced by

$$M = M(t) = 1 + L_0 t$$

or $M = 2$, since $t \in [0, \frac{1}{L_0})$.

(b) The radius r_1 was shown in [5, 6] to be the convergence radius for Newton's method under conditions (12) and (13). It follows from (5) and the definition of r_1 that the convergence radius r of the method (2) cannot be larger than the convergence radius r_1 of the second order Newton's method. As already noted in [5, 6], r_1 is at least as large as the convergence ball give by Rheinboldt [13]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_1$$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L} \to 0.$$

That is our convergence ball r_1 is at most three times larger than Rheinboldt's. The same value for r_R given by Traub [14].

(c) It is worth noticing that method (2) is not changing if we use the conditions of Theorem 1 instead of the stronger conditions given in [1]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [9]

$$\xi = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \qquad \text{for each } n = 0, 1, 2, \dots \tag{32}$$

or the approximate computational order of convergence (ACOC) [9]

$$\xi^* = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \qquad \text{for each } n = 1, 2, \dots \tag{33}$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first Fréchet derivative. Notice that the evaluation of ξ^* does not require that the usage of the solution x^*.

(d) If $s = 0$ and $r = \min\{r_1,\ r_2\}$, then the results of Theorem 1 hold for method (2) replacing method (3) (except (18)).

Numerical example and applications

In this section, we shall check the effectiveness and validity of our theoretical results which we have presented in section 2 on the scheme proposed by Babajee et al. [1]. For this purpose, we shall choose a variety of nonlinear equations and system of nonlinear equations which are mentioned in the following examples including motivational example. At this point, we will choose the following methods

$$\begin{cases} y_n = x_n - \dfrac{2}{3}F'(x_n)^{-1}F(x_n), \\ x_{n+1} = G_{4th\ HM} = x_n - H_1(x_n)A(x_n)F(x_n), \\ z(x_n) = F'(x_n)^{-1}F'(y_n), \\ H_1(x_n) = I - \dfrac{1}{4}\Big(z(x_n) - I\Big) + \dfrac{1}{2}\Big(z(x_n) - I\Big)^2, \\ A(x_n) = \dfrac{1}{2}\Big(F'(x_n)^{-1} + F'(y_n)^{-1}\Big) \end{cases} \tag{34}$$

$$\begin{cases} y_n = x_n - \dfrac{2}{3}F'(x_n)^{-1}F(x_n), \\ z_n = x_n - H_1(x_n)A(x_n)F(x_n), \\ x_{n+1} = z_n - H_2(x_n)A(x_n)F(z_n), \\ H_2(x_n) = 2I - F'(x_n)^{-1}F'(y_n) \end{cases} \tag{35}$$

and

$$\begin{cases} y_n = x_n - \dfrac{2}{3}F'(x_n)^{-1}F(x_n), \\ z_n = x_n - H_1(x_n)A(x_n)F(x_n), \\ w_n = z_n - H_2(x_n)A(x_n)F(z_n), \\ x_{n+1} = w_n - H_2(x_n)A(x_n)F(w_n), \end{cases} \tag{36}$$

having convergence order of $p = 4$, $p = 6$ and $p = 8$ which can be deduced by using $s = 0$, $s = 1$ and $s = 2$, respectively. For computational point of view, we denoted them by M_1, M_2 and M_3, respectively.

First of all, we shall calculate the values of r_R, r_1, r_A, r_2, r_{q_s} and r which are displayed in the Tables 1, 3, 5, 6. So, we can obtain the radius of convergence of the above mentioned methods. Then, we will also verify the theoretical order of convergence of these methods for scalar equations on the basis of the results obtain from computational order of convergence and $\left|\frac{e_n}{e_{n-1}^p}\right|$ (where p is either $p = 4, 6$ or $p = 8$). In the Tables 2, 4 and 7, we displayed the number of iteration indexes (n), approximated zeros (x_n), residual error of the corresponding function $(|F(x_n)|)$, errors $|e_n|$ (where $e_n = x_n - x^*$), $\left|\frac{e_n}{e_{n-1}^p}\right|$ and the asymptotic error constant $\eta = \lim_{n\to\infty}\left|\frac{e_n}{e_{n-1}^p}\right|$. In addition, we calculate the computational order of convergence by using the above formulas (32) and (33). Moreover, we calculate the computational order of convergence, asymptotic error constant and other constants up to several number of significant digits (minimum 1000 significant digits) to minimize the round off error.

In the context of system of nonlinear equations, we also consider a nonlinear system in example 3 to check the proposed theoretical results for nonlinear system. In this regards, we displayed the number of iteration indexes (n), residual error of the corresponding function $(\|F(x_n)\|)$, errors $\|e_n\|$ (where $e_n = x_n - x^*$), $\left\|\frac{e_n}{e_{n-1}^p}\right\|$ and the asymptotic error constant $\eta = \lim_{n\to\infty}\left\|\frac{e_n}{e_{n-1}^p}\right\|$ in the Table 7. Moreover, we use the above mentioned formulas namely, (32) and (33) to calculate the computational order of convergence to further verifying the theoretical order of convergence of nonlinear system.

As we mentioned in the earlier paragraph that we calculate the values of all the constants and functional residuals up to several number of significant digits but due to the limited paper space, we display the values of x_n up to 15 significant digits and the values of other constants namely, $r_R, r_1, r_A, r_2, r_{q_s}$, r, $\xi(COC)$, $\left|\frac{e_n}{e_{n-1}^p}\right|$, η and $\left\|\frac{e_n}{e_{n-1}^p}\right\|$ are up to 5 significant digits. Further, the residual error in the function/system of nonlinear functions ($|F(x_n)|$ or $\|F(x_n)\|$), and the error $|e_n|$ or $\|e_n\|$ are display up to 2 significant digits with exponent power which are mentioned in the following Tables corresponding to the test function. However, minimum 1000 significant digits are available with us for every value.

Furthermore, we consider the approximated zero of test functions when the exact zero is not available, which is corrected up to 1000 significant digits to calculate $\|x_n - x^*\|$. For the computer programming, all computations have been performed using the programming package $Mathematica$ 9 with multiple precision arithmetic. In addition, the meaning of $ae(\pm b)$ is $a \times 10^{\pm b}$ in the tables 1–7.

Example 1 *Let $S = \mathbb{R}, D = [-1, 1], x^* = 0$ and define function F on D by*

$$F(x) = \sin x. \tag{37}$$

Then, we get $L_0 = L = 1$ and $M = 1$. We calculate the different values of the radius of convergence, COC (ξ) etc., which are displayed in the following Tables 1 and 2.

Table 1: Different values of parameters which satisfy Theorem 1

Cases	r_R	r_1	r_A	r_2	r_{q_s}	r
M_1	0.66667	0.44444	0.66667	0.33913	0.19871	0.19871
M_2	0.66667	0.44444	0.66667	0.33913	0.097223	0.097223
M_3	0.66667	0.44444	0.66667	0.33913	0.0401905	0.0401905

Table 2: Convergence behavior of different cases on example 1

Cases	n	x_n	$\|F(x_n)\|$	$\|e_n\|$	ξ	$\left\|\frac{e_n}{e_{n-1}^p}\right\|$	η
M_1	0	0.15	$1.5e(-1)$	$1.5e(-1)$			
	1	$4.22751609372639e(-6)$	$4.2e(-6)$	$4.2e(-6)$		$8.3497e(-3)$	$2.2442e(-7)$
	2	$7.16819456281920e(-29)$	$7.2e(-29)$	$7.2e(-29)$	5.0045	$2.2442e(-7)$	
M_2	0	0.070	$7.0e(-2)$	$7.0e(-2)$			
	1	$-7.42990061829384e(-11)$	$7.4e(-11)$	$7.4e(-11)$		$6.3153e(-4)$	$6.5738e(-13)$
	2	$1.10589328064363e(-73)$	$1.1e(-73)$	$1.1e(-73)$	7.0009	$6.57738e(-13)$	
M_3	0	0.020	$2.0e(-2)$	$2.0e(-2)$			
	1	$-7.56773592959322e(-19)$	$7.6e(-19)$	$7.6e(-19)$		$2.9561e(-5)$	$1.1160e(-21)$
	2	$1.20053454004807e(-166)$	$1.2e(-166)$	$1.2e(-166)$	9.0001	$1.1160e(-21)$	

Example 2 *Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $v = (x, y, z)^T$ and defined F on D by*

$$F(v) = \left(e^x - 1,\ \frac{e-1}{2}y^2 + y,\ z\right)^T. \tag{38}$$

Then the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,\ 0,\ 0)^T$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,\ 1,\ 1\}$, $L_0 = e - 1$, $L = 1.789572397$ and $M = 1.7896$. Hence, we calculate the different values of the radius of convergence, COC (ξ) etc., which are mentioned in the following Tables 3 and 4.

Table 3: Different values of parameters which satisfy Theorem 1

Cases	r_R	r_1	r_A	r_2	r_{q_s}	r
M_1	0.37253	0.15440	0.38269	0.17719	0.067047	0.067047
M_2	0.37253	0.15440	0.38269	0.17719	0.0152413	0.0152413
M_3	0.37253	0.15440	0.38269	0.17719	0.0023526	0.0023526

Example 3 *Let $\mathbb{X} = \mathbb{Y} = C[0,\ 1]$, and consider the nonlinear integral equation of Hammerstein-type defined by*

$$x(s) = \int_0^1 G(s,t)\frac{x(t)^2}{2}dt, \tag{39}$$

Table 4: Convergence behavior of different cases on example 2

Cases, x_0	n	$\|F(x_n)\|$	$\|e_n\|$	ξ	$\left\|\frac{e_n}{e_{n-1}^p}\right\|$	η
$M_1,\ (0.017,\ 0.017,\ 0.018)$	0	$3.0e(-2)$	$3.0e(-2)$			
	1	$1.4e(-7)$	$1.4e(-7)$		0.17667	1.7584
	2	$7.5e(-28)$	$7.5e(-28)$	3.8124	1.7584	
$M_2,\ (0.0022,\ 0.0022,\ 0.0023)$	0	$3.4e(-3)$	$3.4e(-3)$			
	1	$6.4e(-16)$	$6.4e(-16)$		0.19198	5.7481
	2	$4.1e(-91)$	$4.1e(-91)$	5.8845	5.7481	
$M_3\ (0.00022,\ 0.00022,\ 0.00023)$	0	$3.9e(-4)$	$3.9e(-4)$			
	1	$9.9e(-29)$	$9.9e(-29)$		0.19642	18.010
	2	$1.6e(-223)$	$1.6e(-223)$	7.9202	18.010	

where the kernel G is the Green's function defined on the interval $[0,\ 1] \times [0,\ 1]$ *by*

$$G(s,t) = \begin{cases} (1-s)t, & t \le s \\ s(1-t), & s \le t. \end{cases} \tag{40}$$

The solution $x^*(s) = 0$ *is the same as the solution of equation* (1), *where operator* $F : C[0,\ 1] \to C[0,\ 1]$ *is defined by*

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\frac{x(t)^2}{2}dt. \tag{41}$$

Notice that

$$\left\|\int_0^1 G(s,t)dt\right\| \le \frac{1}{8}. \tag{42}$$

Then, we have that the Fréchet- derivative is defined by

$$F'(x)(y(s)) = y(s) - \int_0^1 G(s,t)x(t)dt. \tag{43}$$

So, we get that $F'(x^*(s)) = I$ *and*

$$\left\|F'(x^*)^{-1}(F'(x) - F'(y))\right\| \le \frac{1}{8}\|x-y\|. \tag{44}$$

Hence, we can choose $L_0 = L = \frac{1}{8}$ *and* $M = 2$*. We calculate the different values of the radius of convergence based on the methods, which are mentioned in the following Table 5.*

Table 5: Different values of parameters which satisfy Theorem 1

Cases	r_R	r_1	r_A	r_2	r_{q_s}	r
M_1	5.3333	1.7778	5.3333	1.0014	0.61000	0.61000
M_2	5.3333	1.7778	5.3333	1.0014	0.33700	0.33700
M_3	5.3333	1.7778	5.3333	1.0014	0.16848	0.16848

Example 4 *Returning back to the motivation example at the introduction on this paper, we have* $L = L_0 = 14.5$*,* $M = 2$ *and our required zero is* $x^* = 1$*. We calculate the different values of the radius of convergence, COC* (ξ) *etc., which are given in the following Tables 6 and 7.*

Table 6: Different values of parameters which satisfy Theorem 1

Cases	r_R	r_1	r_A	r_2	r_{q_s}	r
M_1	0.045977	0.015326	0.045977	0.030850	0.002334	0.002334
M_2	0.045977	0.015326	0.045977	0.030850	0.000052082	0.000052082
M_3	0.045977	0.015326	0.045977	0.030850	$1.01954e(-6)$	$1.01954e(-6)$

Table 7: Convergence behavior of different cases on example 4

Cases	n	x_n	$\lvert F(x_n)\rvert$	$\lvert e_n\rvert$	ξ	$\left\lvert\frac{e_n}{e_{n-1}^p}\right\rvert$	η
M_1	0	0.318	$7.3e(-5)$	$3.1e(-4)$			
	1	0.318309886198877	$3.5e(-12)$	$1.5e(-11)$		1636.0	1615.17
	2	0.318309886183791	$2.0e(-41)$	$8.4e(-41)$	4.0007	1615.17	
M_2	0	0.3183	$2.3e(-6)$	$9.9e(-6)$			
	1	0.318309886183791	$1.0e(-25)$	$4.3e(-25)$		$4.5854e(+5)$	$4.5825e(+5)$
	2	0.318309886183791	$6.6e(-142)$	$2.8e(-141)$	6.0000	$4.5825e(+5)$	
M_3	0	0.3183	$2.3e(-6)$	$9.9e(-6)$			
	1	0.318309886183791	$2.8e(-33)$	$1.2e(-32)$		$1.3008e(+8)$	$1.2997e(+8)$
	2	0.318309886183791	$1.2e(-248)$	$5.1e(-248)$	8.0000	$1.2997e(+8)$	

Results and discussion

It is worthy to note that the radii of convergence in the Table 6 are very small. Actually, the radius of convergence depends on the considered function, corresponding bounds and the body of structure of the iterative methods. We can see these things in the above Tables 1, 3 and 5 where we obtain better radii of convergence rather than Table 6. In addition, we also want to check the convergence behavior of the listed methods, when we consider initial approximation out of the convergence domain which can be seen in the Table 6. So, we can say that the proposed iterative methods will always converge to the required root whenever we consider the initial approximation inside of convergence domain however, it is also possible outside the domain. No doubts, the radius of convergence of the listed scheme is decreasing by increasing the number of sub-steps. But, we are getting better and faster convergence towards the required root which can be seen the Tables 2, 4 and 7.

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