Highly Accurate Smoothed Finite Element Methods Based on Simplified Eight-noded Hexahedron Elements

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Abstract

Compared with tetrahedron elements, hexahedron elements are preferred for its high accuracy. However, coordinate mapping required in the hexahedron elements of FEM formulation costs huge running time, leading to poor performance. Besides, the high quality of Jacobian matrix and mesh is required, which affects the accuracy of the strain results greatly. In order to solve these problems, we propose a novel simplified integration technique based on smoothed finite element method (S-FEM) for the eight-noded hexahedron elements, where coordinate mapping is not demanded. The proposed new S-FEM models include: NS-FEM-H8 (using node-based smoothing domains) and FS-FEM-H8 (using face-based smoothing domains). In the work, we divide a non-triangular face segment of a smoothing domain into two triangular sub-segments, since the strain-displacement matrix can be calculated using a summation in S-FEM theory instead of the integration in FEM. Then we conduct the Gauss integration scheme in each triangular face sub-segment in order to avoid the coordinate mapping in quadrilateral face segments. The rest solving algorithm is the same as the standard S-FEM. Through intensive numerical examples, our simplified S-FEM-H8 is approved to have the following features: (1) the strain energy of simplified NS-FEM-H8 is an upper bound of the exact solution; (2) simplified NS-FEM-H8 can overcome the volume locking problems for incompressible materials.

Keywords: eight-noded hexahedron element, coordinate mapping, a novel simplified integration technique, S-FEM-H8.

1. Introduction

Since the late 1950s, researchers have used the finite element method (FEM) [1] as an essential and important tool for the modeling and simulation of practical problems. However, when using FEM we encounter some problems, such as overly stiff issues and the significant loss of accuracy caused by distortions of the mesh.

In recent years, a new numerical method-smoothed finite element method (S-FEM) [2][3] is presented by G.R. Liu to overcome the disadvantages of the standard FEM, which combines the FEM and the mesh-free technique. S-FEM always produces models that are softer than FEM and even softer than the exact model. Besides, S-FEM can obtain relative accurate strain solutions for the distorted mesh where FEM cannot work.

In practical problems, engineers prefer to use linear elements since the linear elements can be easily obtained using automatically dividing program. However, the poor accuracy of the strain solution is the biggest problem for linear elements. Although FEM based on higher-order elements, such as quadrilateral element (Q4) and eight-noded hexahedron element (H8), can overcome the shortcoming of poor precision in lower-order elements, the coordinate mapping and the strict requirement of the quality of the mesh lead to the limitation of the high-order elements.

To solve the problems, we present a novel simplified integration technique for eight-noded hexahedron elements in S-FEM to avoid coordinate mapping. The proposed S-FEM-H8 includes simplified NS-FEM-H8 and simplified FS-FEM-H8. In order to compute the smoothed strain-displacement matrix effectively for smoothing domains, a simplified integration technique is also proposed, in which we divide a non-triangular face segment of a smoothing domain into two triangular sub-segments. Therefore, the surface integration in the S-FEM formulation can be performed without coordinate mapping.

2. Three-dimensional Smoothed Finite Elements

2.1 Discretized Linear Algebraic System of Equations

Standard discretized algebraic system of equations

$$\bar{\mathbf{K}}\bar{\mathbf{d}} = \tilde{\mathbf{f}} \tag{1}$$

where $\bar{\mathbf{d}} \in \mathbb{R}_0^{dN_n}$ is the vector of nodal displacements for the all nodes in the S-FEM model, and $\bar{\mathbf{K}}$ is the smoothed stiffness matrix given in the general form of

$$\bar{\mathbf{K}}_{IJ} = \int_{\Omega} \bar{\mathbf{B}}_I^T \mathbf{c} \bar{\mathbf{B}}_J \mathrm{d}\Omega = \sum_{k=1}^{N_s} \int_{\Omega_k^s} \bar{\mathbf{B}}_I^T \mathbf{c} \bar{\mathbf{B}}_J \mathrm{d}\Omega = \sum_{k=1}^{N_s} \bar{\mathbf{B}}_I^T \mathbf{c} \bar{\mathbf{B}}_J A_k^s \tag{2}$$

the smoothed strain–displacement matrix $\bar{\mathbf{B}}_I$ is computed by Eq. (3)错误!未找到引用源。

$$\bar{\mathbf{B}}_I = \frac{1}{V_k^s} \int_{\Omega_k^s} \mathbf{L}_n(\mathbf{x}) \mathbf{N}(\mathbf{x}) d\Omega = \begin{bmatrix} \bar{b}_{Ix} & 0 & 0 \\ 0 & \bar{b}_{Iy} & 0 \\ 0 & 0 & \bar{b}_{Iz} \\ 0 & \bar{b}_{Iz} & \bar{b}_{Iy} \\ \bar{b}_{Iz} & 0 & \bar{b}_{Ix} \\ \bar{b}_{Iy} & \bar{b}_{Ix} & 0 \end{bmatrix} \tag{3}$$

with

$$\bar{b}_{Ih} = \frac{1}{V_k^s} \int_{\Omega_k^s} n_h(\mathbf{x}) N_I(\mathbf{x}) d\Omega = \frac{1}{V_k^s} \sum_{p=1}^{n_\Omega^s} n_{h,p} N_I(\mathbf{x}_p^G) A_p^{Surf}, \quad (h = x, y, z) \tag{4}$$

The above surface integration along Ω_k^s can be carried out using the Gauss quadrature technique. Where n_Ω^s is the total number of boundary surfaces Ω_k^s and $\mathbf{x}_p^G$ is the Gauss point of the boundary surfaces of Ω_k^s, whose area and outward unit normal are denoted as A_p^{Surf} and $n_{h,p}$, respectively.

When the displacement field along the boundary Ω_k^s is used, four Gauss points are in demand to a quadrilateral surface segment. The quadrilateral segment has in general an arbitrary shape in the physical coordinate system. When the general procedure is used to create the shape functions for Q4 elements, we will meet the problems with the compatibility issue. Thus, the formulation of H8 elements for 3D problems requires a coordinate mapping procedure which is a cost procedure.

However, for triangular surface segment, one Gauss point is sufficient for linear element. And using directly the physical Cartesian coordinate system Oxy, three shape functions have the following forms:

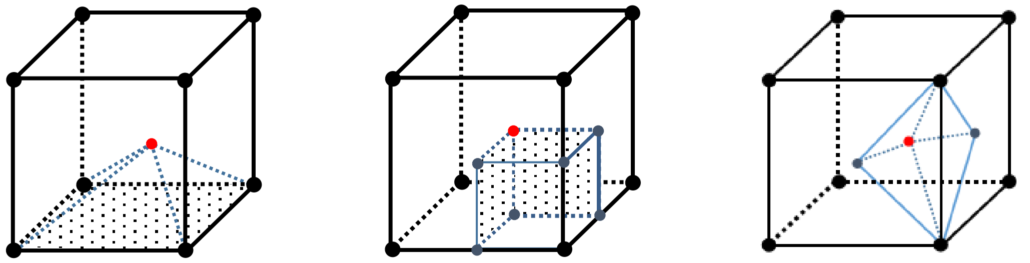
$$N_j = \frac{1}{6A_p^{Surf}}\left(a_j + b_j x + c_j y\right) \tag{5}$$

in which subscript *j* varies from 1 to 3, A_p^{Surf} is the area of triangular surface segment, and a_j, b_j, c_j are constants. It is clear that the strain-displacement matrix $\bar{\mathbf{B}}_I$ for the above segment is a constant matrix. And any coordinate mapping is not demanded.

Hence, we try to employ the advantage of the triangular surface segment to set up a simplified S-FEM-H8.

2.2 3D Simplified FS-FEM-H8 and NS-FEM-H8

In 3D problems, face-based smoothing domain is the simplest smoothing domain, which is created by joining the center points of the two adjacent elements to the four nodes of the face, shown in Figure 1(a). Node-based smoothing domain is a little complex, which is constructed by successively connecting the middle points of the edges connected to this node and the centroids of the faces containing this node, and the centroids of the faces and the center point of the element, shown in Figure 1(b). Edge-based smoothing domain is formed by sequentially connecting the center point of the element to the two nodes of this edge, the centroids of the faces containing the edge to the two nodes of this edge and the centroids of the faces containing the edge to the center point of the element, shown in Figure 1(c).



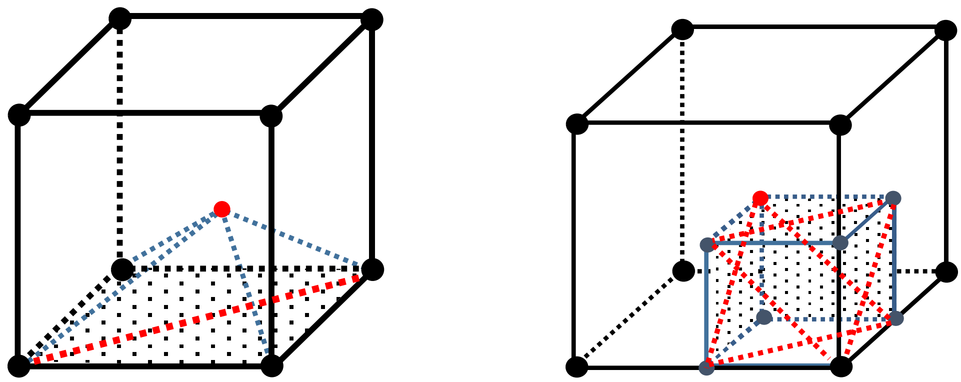
(a) (b) (c)

Figure 1 (a) a face-based smoothing domain (b) a node-based smoothing domain (c) a edge-based smoothing domain

From Figure 1, it is easily found that the smoothing domain has quadrilateral surface segments and triangle surface segments. For quadrilateral surface segments, the coordinate mapping is needed to obtain Jacobian matrix when the Gauss integration scheme implements in Eq.

(4). For triangular surface segments, such as the surface segments of edge-based smoothing domain and triangular surface segments of face-based smoothing domain, because the strain-displacement matrix is a constant matrix, no numerical integration is needed to compute the elemental stiffness matrix.

Therefore, in order to avoid the coordinate mapping, we divide a non-triangular surface segment of a smoothing domain into two triangular sub-segments to form simplified FS-FEM-H8 and simplified NS-FEM-H8, shown in Figure 2. For standard ES-FEM-H8, the smoothing surface segments are all triangles, so that we still use the standard ES-FEM-H8.

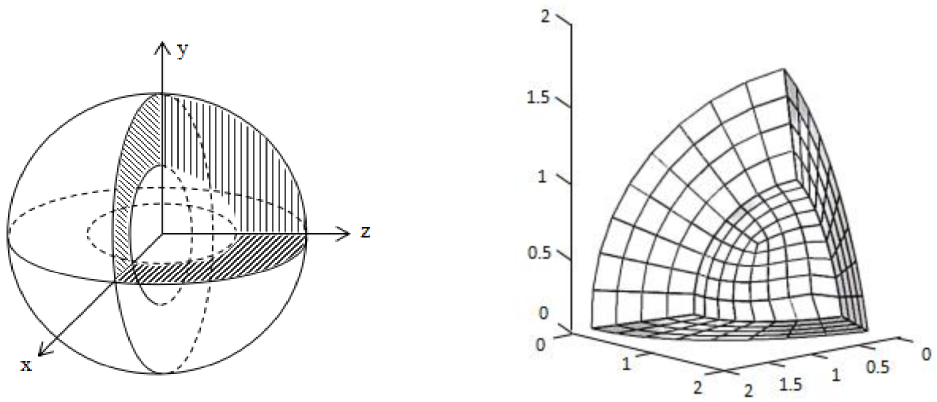


(a) **(b)**

Figure 2 (a) Simplified face-based smoothing domains (FS-FEM-H8) (b) Simplified node-based smoothing domains (NS-FEM-H8)

3. Numerical Example

Consider a 3D Lame problem consists of a hollow sphere with inner radius a=1m, outer radius b=2m and subjected to an internal pressure P=1 N/m^2. As the problem is spherically symmetrical, only one-eighth of the sphere model is shown in Figure 3 and symmetry conditions are imposed on the three symmetric planes.



(a) (b)

Figure 3 (a) hollow sphere problem domain and (b) one-eighth model discretized using eight-noded hexahedron elements.

Figure 4 shows that the convergence of the strain energy solution obtained using different methods for the hollow sphere subjected to inner pressure. We can easily find that the strain energy of simplified FS-FEM-H8 and NS-FEM-H8 are both more accurate than that of FEM-H8. Besides, it confirms the upper bound property on the strain energy of simplified NS-FEM-H8 and the lower bound property of simplified FS-FEM-H8 and FEM-H8 for this 3D problem. Also, the distribution of the radial displacement, radial and tangential stresses using FEM-H8, simplified FS-FEM-H8, simplified NS-FEM-H8 and standard ES-FEM-H8 compared with the analytical solution is presented in Figure 5.

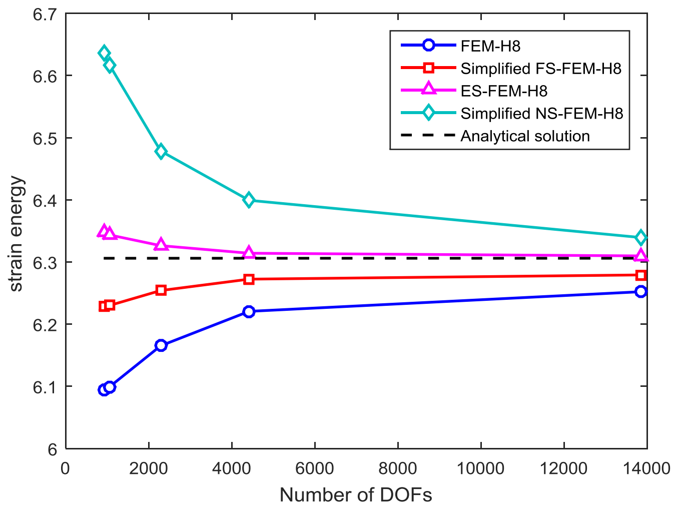
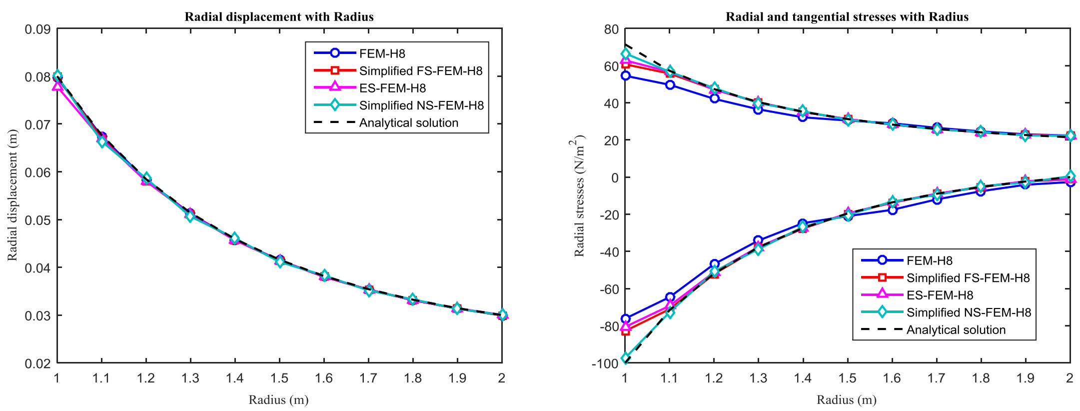


Figure 4 Convergence of the strain energy solution obtained using different methods for the hollow sphere subjected to inner pressure.



(a) (b)

Figure 5 (a) Radial displacement and (b) radial and tangential stresses for the hollow sphere subjected to inner pressure.

Figure 6 plots the error in the displacement norm against Poisson's ratio changing from 0.4 to 0.4999999 obtained using eight-noded hexahedron elements. The results show that simplified NS-FEM-H8 is naturally immune from volumetric locking, while standard FEM-H8 is subjected to volumetric locking, resulting in a drastic accuracy loss in the numerical solutions.

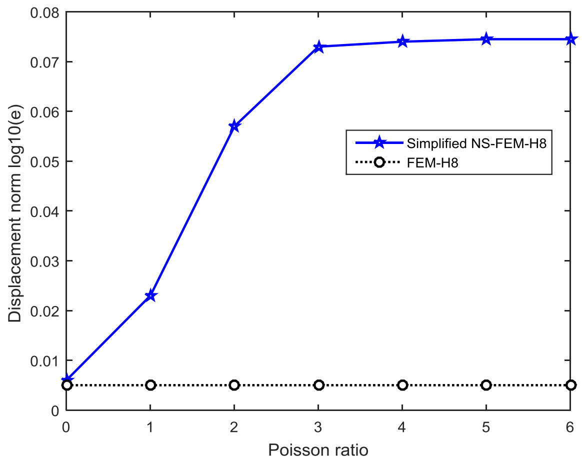


Figure 6 Displacement norm versus different Poisson's ratios for the hollow sphere subjected to inner pressure.

4. Conclusion

In the paper, we put forward a simplified NS-FEM-H8 and ES-FEM-H8 to avoid coordinate mapping for 3D problems, which can improve the efficiency of the algorithm. At the same time, the novel methods can maintain the high accuracy of the higher-order elements. Through numerical examples, some main conclusions are presented as follows:

● The strain energy of simplified NS-FEM-H8 is an upper bound of the exact strain energy, while the solutions of simplified FS-FEM-H8 and standard FEM-H8 are upper bound of the exact strain energy.

● For the almost incompressible problems, simplified NS-FEM-H8 has the characteristic of simplified integration that is free from volume locking.

References

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