A robust inversion-based Fourier Transformation algorithm used in the interpretation of non-equidistantly measured magnetic data

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Abstract

Fourier transformation is one of the most frequently used operation in data processing. In case of discrete data sets the Discrete Fourier Transformation (DFT) algorithm is often applied. As the measured data always contain noise, the noise sensitivity of the processing methods is an important feature. The noise registered in the time domain is directly transformed into the frequency domain. Therefore the traditional discrete variants of Fourier transformation are very noise sensitive procedures. In the field of inverse problem theory a variety of numerous procedures are available for noise rejection, so if the Fourier transformation is formulated as an inverse problem these tools can be used to reduce the noise sensitivity.

It is well-known from inverse problem theory that simple least square methods give optimal results only when data noises follow Gaussian distribution. The practice of geophysical inversion shows that the least square solutions are very sensitive to sparsely distributed large errors, i.e. outliers in the data set and the estimated model parameters may even be completely non-physical. There are various ways to address the question of statistical robustness: the Least Absolute Deviation (LAD) method (minimizing the L1 norm of the misfit between the observed and predicted data) and the Cauchy weighted Iteratively Reweighted Least Squares method are well-known. A more flexible method can be defined by modifying the weights with the help of Steiner's Most Frequent Value method. In the present paper the 2D Fourier transformation is handled as robust inverse problem using IRLS algorithm with Cauchy-Steiner weights. The discretization of the continuous Fourier spectra is given by a series expansion with the scaled Hermite functions as basis functions. The expansion coefficients are determined by solving an overdetermined inverse problem. In order to define a quick algorithm in calculating the Jacobi matrix of the problem, the special feature that the Hermite functions are eigenfunctions of the Fourier transformation was used. It is shown that the method can successfully be applyed in the interpretation of geomagnetic data sets measured in 2D arrays. There is an important feature of the new inversion based Fourier Transformation: the measurement array should not be equidistant along the (x,y) directions. It will be shown in the presentation that the method gives accurate results even in the interpretation of geomagnetic data measured also in "random walk" measurement array.

Key words: Fourier Transformation, Inversion, Series Expansion, Cauchy Noise, Random walk measurement.

Introduction

The systematic improvement in geophysical data acquisition over the years demand for more innovative data processing methods. Traditional survey designs employ equidistant measurement on a regular grid. Unfortunately, measurements are sometimes taken out of grid due to several obstacles encountered in the field of survey. This has necessitated the development of methods for the effective processing of equal in random-walk geophysical measurements. Fourier transformation is one of the most frequently used operation in data processing. It connects the time domain of signal registration and the frequency domain of signal processing. In case of discrete data sets, the DFT algorithm is often applied (Discrete Fourier Transformation). As the measured data always contain noise, the noise sensitivity of the processing methods is an important feature. The noise registered in the time domain is directly transformed into the frequency domain. Therefore, the traditional discrete variants of Fourier transformation are very noise sensitive procedures. Dobróka et al. (2015) presented an inversion based 1D Fourier transformation method (S-IRLS-FT) which proved to be an effective tool for noise reduction. The method was generalized to 2D, and an application is presented in solving reduction to the pole of the magnetic data set (Dobróka et al., 2017). In this paper, it is shown that the newly developed inversion-based Fourier transformation algorithm can also be used in processing non-equidistant (even in random walk) measurement geometry dataset.

Method Development (2D inversion-based Fourier transformation)

The 2D Fourier transform of a function *u(x,y)* can be calculated by the integral

$$U(\omega_x,\omega_y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x,y)\,e^{-j(\omega_x x+\omega_y y)}dx\,dy\,, \tag{1}$$

its inverse is given by the formula

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(\omega_x,\omega_y)\,e^{j(\omega_x x+\omega_y y)}d\omega_x d\omega_y\,, \tag{2}$$

where x, y are the spatial coordinates, $U(\omega_x,\omega_y)$ means the 2D spatial-frequency spectrum and ω_x, ω_y indicate the spatial-angular frequencies. Following a powerful inversion strategy (the so-called series expansion-based optimization) developed at the Geophysical Department of the University of Miskolc (Gyulai and Szabó 2014, Dobróka et al. 2015), the discretization of the continuous spectrum is written in the form of a series expansion,

$$U(\omega_x,\omega_y)=\sum_{n=1}^{N}\sum_{m=1}^{M}B_{n,m}\,\Psi_{n,m}(\omega_x,\omega_y)\,, \tag{3}$$

where $\Psi_{n,m}(\omega_x,\omega_y)$ are frequency dependent basis functions, $B_{n,m}$ are the expansion coefficients which represent the model parameters of the inverse problem. The basis function system should be square integrable in the interval (-∞, ∞). The Hermite functions meet this criterion with an additional advantage. The determination of Jacobian matrix in an inversion process requires the calculation of a complex integral for the interval (-∞, ∞), which is usually a time-consuming process. As it was derived by Dobróka et al. (2015) the elements of the Jacobian matrix can be considered as the inverse Fourier transformation of the basis function system. Therefore, they can be calculated more easily if the basis functions are chosen from the eigenfunctions of the inverse Fourier transformation. It can be shown, that the normed and scaled Hermite functions

$$H_n(\omega_x,\alpha)=\frac{e^{-\frac{\alpha\omega_x^2}{2}}h_n(\omega_x,\alpha)}{\sqrt{\sqrt{\frac{\pi}{\alpha}}\,n!(2\alpha)^n}}\,,\qquad \text{where}\quad h_n(\omega_x,\alpha)=(-1)^n e^{\alpha\omega_x^2}\left(\frac{d}{d\omega_x}\right)^n e^{-\alpha\omega_x^2}\,, \tag{4}$$

$$H_m(\omega_y,\beta)=\frac{e^{-\frac{\beta\omega_y^2}{2}}h_m(\omega_y,\beta)}{\sqrt{\sqrt{\frac{\pi}{\beta}}\,m!(2\beta)^m}}\,,\qquad \text{where}\quad h_m(\omega_y,\beta)=(-1)^m e^{\beta\omega_y^2}\left(\frac{d}{d\omega_y}\right)^m e^{-\beta\omega_y^2}\,, \tag{5}$$

are eigenfunctions of the inverse Fourier transformation and the Jacobian matrix of the inverse problem can be written as

$$G_{k,l}^{n,m} = \frac{(j)^{n+m}}{\sqrt[4]{\alpha\beta}} H_n^{(0)}\left(\frac{x_k}{\sqrt{\alpha}}\right) H_m^{(0)}\left(\frac{y_l}{\sqrt{\beta}}\right). \tag{6}$$

Here $H_n^{(0)}, H_m^{(0)}$ denote the non-scaled Hermite functions. Applying Eq. (6) provides a fast solution to the forward problem

$$u(x_k, y_l) = \sum_{n=1}^{N}\sum_{m=1}^{M} B_{n,m} G_{k,l}^{n,m}. \tag{7}$$

The introduction of single indices instead of double ones ($s=k+(l-1)K$, $i=n+(m-1)N$) makes the calculations much easier

$$u(x_k, y_l) = u_{k,l} = u_s, \quad B_{nm} = B_i, \quad u_s = \sum_{i=1}^{I} B_i G_{s,i}, \quad (i = 1,\ldots,I,\ s = 1,\ldots,S), \tag{8}$$

where $I=N+(M-1)N=NM$ is the number of series expansion coefficients, $S=K+(L-1)N=KL$ is the maximal number of data. With these simplifications, the deviation of measured and calculated data (e_s) can be calculated as

$$e_s = u_s^{(measured)} - \sum_{i=1}^{I} B_i G_{s,i}. \tag{9}$$

The Iteratively Reweighted Least Squares (IRLS) method is a reliable data processing procedure if the data set contains outliers. It can be combined with Cauchy weights, where the scale parameters σ^2 have to be known a priori. Using the Most Frequent Value method (Steiner 1997), Szegedi and Dobróka (2014) proposed the use of Steiner weights because the scale parameters are derived from the statistical sample in an inner iteration cycle. To make the Fourier transformation more robust, a new Steiner-weighted IRLS method (S-IRLS) was introduced where the following weighted norm is minimized

$$E_w = \sum_{s=1}^{N} W_{ss} e_s^2. \tag{10}$$

where W_{ss} is the s-th Steiner weight. The normal equation for the j-th IRLS step can be written as

$$\underline{\underline{\mathbf{G}}}^{\mathbf{T}} \underline{\underline{W}}^{(j-1)} \underline{\underline{\mathbf{G}}} \vec{B}^{j} = \underline{\underline{\mathbf{G}}}^{\mathbf{T}} \underline{\underline{W}}^{(j-1)} \vec{u}^{(measured)}. \tag{11}$$

After reaching the stop criteria, the series expansion coefficients determined in the last iteration step are considered the solution of the problem.

Application

To prove the applicability of the 2D robust inversion based S-IRLS FT method, it was tested on synthetic magnetic data sets: one was noise-free and the other contained random noise following Cauchy distribution. 1089 measurement points were assumed along with a 5 m x 5 m grid which was further randomized to obtain non-equidistant measurements. Data were generated for a surface between -100 m, and 100 m both in the x and y directions above a 'C_L' shaped magnetic body (inclination I=63°, declination D=3°, magnetization 200 nT). The surface magnetic data were calculated by the Kunaratnam (1981) method and was subsequently reduced to the pole (I=90°) by applying the formula in the frequency domain

$$R(u,v) = T(u,v)S(u,v), \tag{12}$$

where $T(u,v)$ is the 2D Fourier transform of the magnetic data set, $S(u,v)$ is the frequency domain operator of pole reduction and $R(u,v)$ is the reduced data set after the data reduction process. First, the reduction to the pole was performed by using conventional DFT algorithm. The map of noiseless magnetic data on equidistant grid and its reduced to pole are given in Fig. 1a and Fig.1b, respectively. The data was contaminated with Cauchy noise to produce Figure 2a which was reduced to pole using the traditional DFT method. The poor noise reduction capability of the DFT method can be clearly seen in Figure 2b. This underlines the need for a new, robust and outlier-resistant Fourier Transformation method.

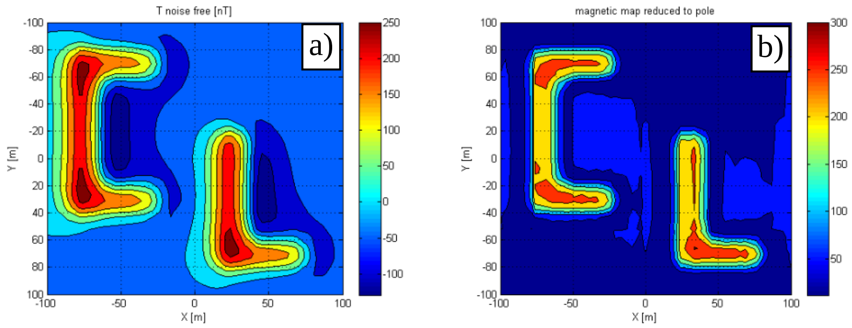

Figure 1 *Noise-free data set and the conventional DFT. a) Magnetic map without pole reduction. b) Magnetic map reduced to the pole using conventional DFT.*

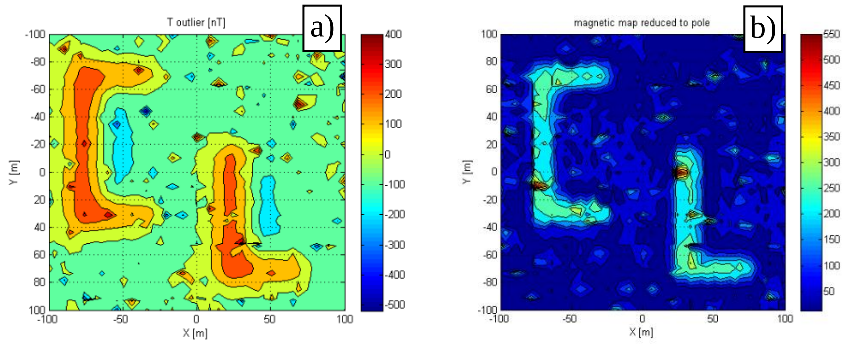

Figure 2 *Data set with Cauchy noise and the conventional DFT. a) Magnetic map without pole reduction. b) Magnetic Map reduced to the pole using conventional DFT.*

Therefore, the magnetic data were processed using the new S-IRLS-FT algorithm instead of traditional DFT. The same map of noisy data (Fig. 3a) and its processing result are presented (Fig. 3b). Comparing Fig. 2b (the pole reduced map on noisy data set using conventional DFT) and Fig. 3b (the pole reduced map on noisy data set using S-IRLS-FT), sufficient improvement can be seen in the processed data using S-IRLS-FT method. The traditional DFT left more spikes of Cauchy noise after the pole reduction causing artifacts and a possible misinterpretation of magnetic anomalies. The high noise reduction capability of the new S-IRLS-FT algorithm is clearly observable.

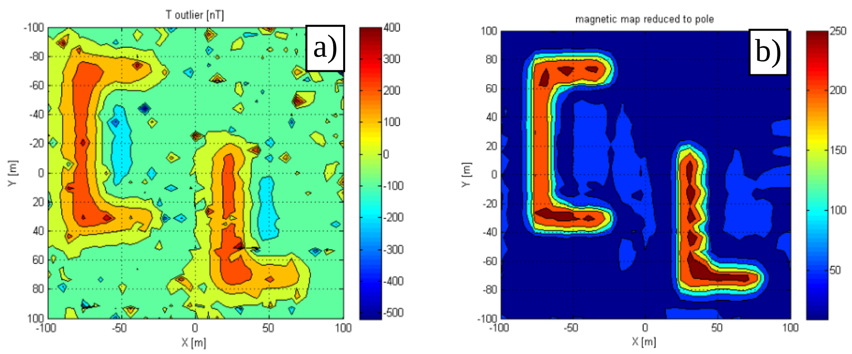

Figure 3 *Data set with Cauchy noise and the new S-IRLS-FT. a) Magnetic map without pole reduction. b) Magnetic map reduced to the pole using the new S-IRLS-FT.*

As an important feature of the new inversion-based Fourier Transformation, the measurement array should not be equidistant along the (x,y) directions. This is demonstrated in Fig. 4a and 4b. In Fig. 4a, (produced from Fig. 1a and same as Fig. 1b) the arrangement of the measurement points was equidistant. To generate a highly non-equidistant arrangement, we shifted randomly all the measurement point from their “regular” place in the sampling interval before the S-IRLS FT method was applied. The result in the noise-free case is shown in Fig. 4b., which is similar to Fig. 4a. demonstrating that the method gives accurate results even in “random walk” measurements.

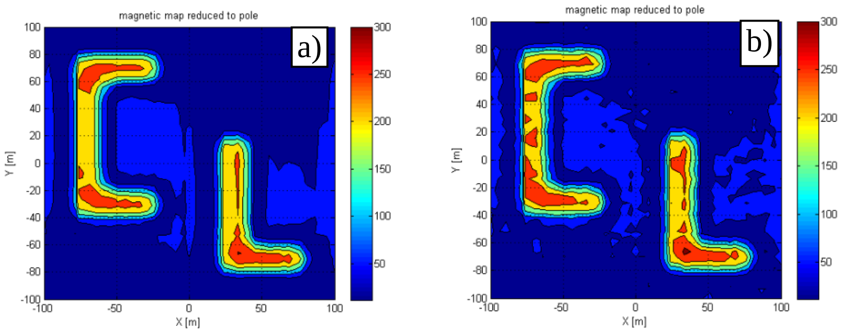

Figure 4 *Reduction to the pole of noise-free data set. a) Equidistant measurement system. b) Random walk measurement system.*

Conclusion

The Discrete Fourier Transformation (DFT) is a common data processing method but incorporate some level of noise in the transformation process. The introduced 2D S-IRLS-FT method treats the Fourier transformation as an inverse problem. The spectrum is discretized by series expansion and the inversion problem is solved for the series expansion coefficients by IRLS method using Steiner weights. Taking advantage of the good features of Hermite functions described in this paper, they were chosen as basis functions making the algorithm quicker. Comparatively, the newly introduced inversion-based Fourier transformation algorithm (2D S-IRLS-FT method) has a higher noise rejection capability than the traditional DFT Method as demonstrated in reduction to pole of magnetic data. In this paper, it was further shown that the inversion-based Fourier transformation algorithm can be effectively used in processing data set collected in non-equidistant (even in random walk) measurement geometry.

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