Comparison of Characteristic-wise WENO and Central Difference Schemes With Numerical Viscosity Models for the Unsteady Compressible Flow

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Abstract

In the computation of unsteady compressible fluid dynamics, unnecessary numerical oscillations can appear in the domain near the sharp discontinuities such as shock or slip line, especially for the high-order spatial accuracy. The WENO(weighted essentially no oscillation) method can delete the oscillation of numerical dissipation error, and so can the central difference methods with artificial viscosity, which are very economical for the computational cost. The characteristics of conventional WENO and central difference schemes are compared with each other for a benchmark problem in this study where LF(Lax-Friedrichs) and Garnier filters are used with ACM(Artificial Compression Method) switch acting near the sharp-gradient discontinuities.

Keywords: **Compressible Flow, WENO, Central Difference, Numerical Filter, ACM Switch, Numerical Viscosity**

Introduction

Originally, the discontinuity of signal in the physical property can produce improper oscillation due to a loss of information in the sampling of continuous analog data, which can be analyzed as the truncation of high-order Fourier series terms even in the various experiments. In the similar principle, the truncation error in Taylor series expansion generates numerical oscillation for the finite difference approximation. For the spatial accuracy more than the second order, a dissipation error is inevitable in the central difference schemes. In the WENO schemes[1] developed by Jiang and Shu, the smooth indicator acts as a sensor for the gradient, giving a filtered solution like upwind or TVD(total variation diminishing) methods. However, this kind of schemes sacrifices an order of spatial accuracy that is an odd number: for example, three, five, and seven, etc. The central difference schemes with ACM sensors[2] developed by Yee et al. has shown a possibility to develop a filtered method with filters, and various filter models are proposed such as Garnier et al.[3] and Kim and Kwon [4]. Generally, so far the central-difference based methods show better result in the smooth region like vortex and slip layer, but the numerical oscillation can be very critical in the discontinuous waves. Therefore, systematic approach for the selection of schemes and filters is required in the development of numerical codes. In this study, we compared characteristics and performances related with three numerical schemes for the numerical simulation of unsteady compressible flow: WENO, central difference with Garnier filter, and central difference with LF filter.

Numerical Schemes

In this section, three numerical methods are explained, and the Euler equations is expressed for the conservative dependent variable vector $\boldsymbol{Q} = [\rho, \rho u_k, E]^T$:

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \frac{\partial \boldsymbol{F}_k}{\partial x_k} = 0 \tag{1}$$

where the flux is $\boldsymbol{F}_k = \left[\rho u_k, \rho u_k u_j + p\delta_{kj}, u_k(E+p)\right]^T$, and the equation of state is $p = (\gamma - 1)\left(E - \rho u_k u_j\right)/2$.

Eq. (1) is semi-discretized in space with the third-order Runge-Kutta algorithm for temporal integration[1]:

$$\frac{\Delta \boldsymbol{Q}_j}{\Delta t} = L\left(\boldsymbol{Q}_j^n\right) = -\frac{1}{2}\left(\boldsymbol{F}_{j+\frac{1}{2}}^n - \boldsymbol{F}_{j-\frac{1}{2}}^n\right) \tag{2}$$

where the time step $\boldsymbol{\Delta t}$ is restricted by CFL(Courant-Friedrichs-Lewy) condition. The numerical flux in Eq. (2) is computed with WENO and central difference methods.

WENO Scheme

The CW(characteristic-wise)-WENO method consists of the following algorithm[1]:

1. Projection to the characteristic field
2. Lax-Friedrichs flux splitting
3. WENO reconstruction
4. Transform back into physical projection

The left eigen-vector matrix from the Roe-averaged Jacobian matrix at the right face of the computational cell[5], $\boldsymbol{R}_{j+\frac{1}{2}}^{-1}$ is used for the projection to the characteristics field.

$$\boldsymbol{q}_l = \boldsymbol{R}_{j+\frac{1}{2}}^{-1}\boldsymbol{Q}_l \tag{3}$$

$$\boldsymbol{f}_l = \boldsymbol{R}_{j+\frac{1}{2}}^{-1}\boldsymbol{F}_l \tag{4}$$

Using Lax-Friedrichs flux splitting, the maximum eigen-value is calculated for the region of influence in the hyperbolic partial differential equation system, Eq. (1). The WENO interpolation applies the convex sum of weighted average in the numerical stencil. The weights are coefficients expressed as a function of smooth indicators and optimal coefficients for the finite difference. The weighted combination of ENO flux results in odd numbered order of spatial accuracy. In the last process, Eqs. (3-4) are transformed back to the primitive variables. The scheme is implemented as a fifth order of spatial accuracy in the present study.

Central Difference Schemes

The central difference flux is applied in Eq. (2), and it is filtered at the last procedure in each time marching:

$$\boldsymbol{Q}^{n+1} = \widehat{\boldsymbol{Q}}^{n+1} + \Delta t L_f\left(\widehat{\boldsymbol{Q}}^{n+1}\right) \tag{5}$$

where L_f is the spatial operator applying a low numerical viscosity filter, which can be controlled selectively for the large-gradient region with ACM switch function[2].

The numerical viscosity that is very similar with a flux in Eq. (5), $\boldsymbol{\phi}$ can be modelled with various methods. Garnier filter[3] subtracts the central difference flux component, $\hat{\boldsymbol{f}}^0$ from the WENO characteristic flux, which is similar with upwind method.

$$\boldsymbol{\phi}_{i+\frac{1}{2}} = \hat{\boldsymbol{f}}^+_{i+\frac{1}{2}} + \hat{\boldsymbol{f}}^+_{i-\frac{1}{2}} - \hat{\boldsymbol{f}}^0_{i+\frac{1}{2}} \tag{6}$$

Kim and Kwon[4] designed a new filter based on the Lax-Friedrichs numerical viscosity using characteristic-wise WENO method as a filter.

$$\boldsymbol{\phi}_{i+\frac{1}{2}} = -\frac{1}{2}\lambda_{max}\left(\boldsymbol{Q}^R_{i+\frac{1}{2}} - \boldsymbol{Q}^L_{i+\frac{1}{2}}\right) \tag{7}$$

where λ_{max} denotes the maximum eigen values.

Convergence Test

Eq. (1) under an initial condition, $\rho_0 = 1 + 0.2\sin(\pi x)$ and $u_0 = p_0 = 1$ is solved for the convergence test with three methods described in the previous section. Errors and orders are given in Table 1 for the number of cells, N in the domain of $x\epsilon[-1,1]$. The order of accuracy satisfies fifth order at N=160 for all schemes.

Table 1. Result of convergence test

Numerical Method	N	L_1 error	L_∞ error	L_1 order	L_∞ order
WENO5	10	1.222E-02	1.757E-02	-	-
	20	6.495E-04	1.002E-03	4.23	4.13
	40	2.075E-05	3.735E-05	4.97	4.75
	80	6.479E-07	1.198E-06	5.00	4.96
	160	2.017E-08	3.640E-08	5.01	5.05
Central Difference with Garnier Filter	10	3.915E-03	5.377E-03	-	-
	20	1.060E-04	1.657E-04	5.21	5.02
	40	2.500E-06	5.387E-06	5.41	4.94
	80	3.948E-08	1.338E-07	5.98	5.33
	160	1.151E-09	3.612E-09	5.10	5.21
Central Difference with LF Filter	10	3.915E-03	5.377E-03	-	-
	20	3.072E-04	5.085E-04	3.67	3.40
	40	9.784E-06	1.955E-05	4.97	4.70
	80	3.058E-07	5.760E-07	5.00	5.09
	160	8.942E-09	1.504E-08	5.10	5.26

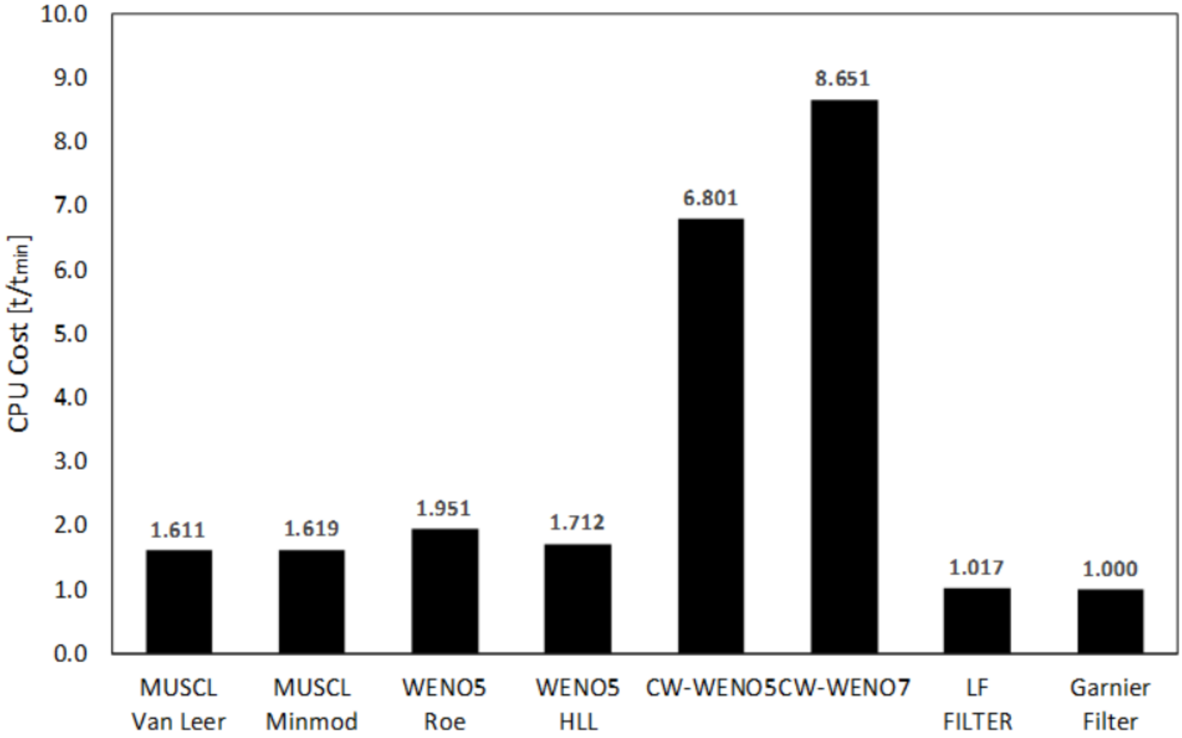

Figure 1. Comparison of CPU times

Computational Cost

The computational cost for the same problem with the same number of grids is compared in Fig. 1, which gives data from various schemes. The minimum time cost is achieved from the central difference with Garnier filter. It shows that the LF filter consumes only 1.7% computational time more than the minimum, but, however, the characteristic-wise WENO consumes about 6.8 times of time because it must perform the matrix inversion to transform back from Eq. (3-4). The time cost of WENO increases about 6.8 times because it should perform the matrix inversion at each intermediate Runge-Kutta time step, and the performance of LF filter is almost similar with Garnier model.

Summary

Three numerical methods are implemented for the computation of unsteady compressible flow: characteristic-wise WENO and central difference method with Garnier and LF filters. From the convergence test, all of them achieved the fifth-order spatial accuracy. The Garnier filter shows the best performance in the convergence test and the CPU cost. However, the test for more complicated problem related with shock and discontinuity waves can differ from the present simple benchmark problem.

Acknowledgement

This work is financially supported by NRF (National Research Foundation, 2016R1D1A3-B01015543) and KETEP (Korea Institute of Energy Technology Evaluation and Planning, No. 20174010201350).

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