A global sensitivity analysis method for multi-input multi-output system

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Abstract
Commonly, variance-based global sensitivity analysis methods are popular and applicable to quantify the impact of a set of input variables on output response. However, for many engineering practical problems, the output response is not single but multiple, which makes some traditional sensitivity analysis methods difficult or unsuitable. Therefore, a novel global sensitivity analysis method is presented to evaluate the importance of multi-input variables to multi-output responses. First, assume that a multi-input multi-output system (MIMOS) includes $n$ variables and $m$ responses. A set of summatory functions $G(x)$ and $H(x)$ are constructed by the addition and subtraction of any two response functions. Naturally, each response function is represented using a set of summatory function. Subsequently, the summatory functions $G(x)$ and $H(x)$ are further decomposed based on the high dimensional model representation (HDMR), respectively. Due to the orthogonality of all the decomposed function sub-terms, the variance and covariance of each response function can be represented using the partial variances of all the decomposed function sub-terms on the corresponding summatory functions, respectively. The total fluctuation of MIMOS is calculated by the sum of the variances and covariances on all the response functions. Further, the fluctuation is represented as the sum of the total partial variances for all the $s$-order function sub-terms and the total partial variance is the sum of $n$ partial variances for the corresponding $s$-order function sub-terms. Then, the function sensitivity index (FSI) $FSI_s$ for $s$-order function sub-terms is defined by the ratio of the total partial variance and total fluctuation, which includes first-order, second-order, and high-order FSI. The variable sensitivity index (VSI) $VSI_{x_i}$ of variable $x_i$ is calculated by the sum of all the FSIs including the contribution of variable $x_i$. Finally, numerical example and engineering application are employed to demonstrated the accuracy and practicality of the presented global sensitivity analysis method for multi-input and multi-output system.

Keywords: Global sensitivity analysis (GSA), High dimensional model representation (HDMR), Variance and covariance decomposition, multi-input multi-output system (MIMOS)

1 Introduction

Sensitivity analysis (SA) is an effective tool for quantifying the influence of input variables to output response and can identify the influential variables and help designers clearly recognize what should be of great concern [1, 2]. It has been widely used in many engineering practical problems, such as structural design [3, 4], parameter identification [5]. In order to solve different types of engineering problems, researches on sensitivity analysis methods have been continuously promoted. Generally, SA is classified into two categories: local sensitivity analysis (LSA) and global sensitivity analysis (GSA). LSA methods are usually used to evaluate the local effect on output response when the small perturbation occurs on the nominal value of input variable. The main disadvantages of LSA are that it does not account...
for interactions between parameters. GSA as a general and comprehensive approach takes into account all the variation range of the parameters in entire range space. It aims at determining which input variables have more important influence on the output response and shows the ranking of influential variables [6, 7]. The frequently-used GSA methods include regression methods, screening-based methods, variance-based methods, and meta-model methods, and so on [7, 8]. Although Sobol’ method based on variance decomposition is more popular among these GSA methods, its interactions cannot be further decomposed, which makes the ranking of influential variables highly likely to be wrong, especially strong non-linear system [7-10]. To further decompose the effects of interactions, Liu et al. [7] improved the Sobol’ method by the combination of variance decomposition and partial derivation integral. However, these GSA methods that are only effective for a multi-input single output system (MISOS) are not suitable for a multi-input multi-output system (MIMOS).

Recently, it has attracted wide attention in terms of SA to MIMOS. Scholars have done some work. Gamboa et al. [11] defined a generalization of Sobol sensitivity indices based on the trace of covariance matrix for multi-output system. Cheng et al [12] developed a multivariate output GSA method by employing multi-output support vector regression (M-SVR). Xu et al. [13] proposed a mapping-based hierarchical sensitivity analysis method to calculate sensitivity indices of multilevel systems with multidimensional correlations. Nevertheless, these multivariate output GSA methods are used to assess the effects of multi-inputs to multi-outputs only considering the contribution of variances but ignoring the contribution of covariances. Hence, this paper presents a novel GSA method for MIMOS comprehensively considering the influence of variance and covariance.

2 Global Sensitivity Analysis Method for Multi-inputs Multi-outputs

2.1 Variance Decomposition

Consider a square-integrable-function \( f(x) \) defined in the unit hypercube \( R^n = \{x | 0 \leq x_i \leq 1, i=1,2,\ldots,n \} \). The performance function decomposition can be given as:

\[
    f(x) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{1 \leq i < j \leq n} f_{ij}(x_i,x_j) + \ldots + f_{12\ldots n}(x_1,x_2,\ldots,x_n)
\]  

(1)

If the integral with respect to its own variable is zero for any function term of HDMR, namely

\[
    \int f_{i} (x_i) \, dx_i = 0, \quad k = i_1, i_2, \ldots, i_m, \quad 1 \leq i_1 < i_2 < \ldots < i_m \leq n
\]  

(2)

The Sobol’ function sub-terms can be uniquely determined as follows:

\[
    f_0 = \int f(x) \, dx
\]  

(3)

\[
    f_i (x_i) = \int f(x) \prod_{k \neq i} dx_k
\]  

(4)

\[
    f_{ij}(x_i,x_j) = \int f(x) \prod_{k \neq i,j} dx_k
\]  

(5)

Any two different Sobol’ functions satisfy the orthogonal condition as follows,

\[
    \int f_{i_1\ldots i_m}(x_{i_1},x_{i_2},\ldots,x_{i_m}) f_{j_1\ldots j_k}(x_{j_1},x_{j_2},\ldots,x_{j_k}) \, dx = 0.
\]  

(6)

The variance decomposition can be given by integrating \( f^2(x) \),

\[
    D = \sum_{i=1}^{n} D_i + \sum_{1 \leq i < j \leq n} D_{ij} + \ldots + D_{12\ldots n}
\]  

(7)

The sensitivity index and total sensitivity index is calculated respectively

\[
    S_{\hat{D}} = \frac{D_{\hat{D}}}{D}, \quad S_{\hat{D}^2} = 1 - S_{\hat{D}}, \quad 1 \leq i_1 < i_2 < \ldots < i_m \leq n.
\]  

(8)
2.2 Multivariate Output Sensitivity Indices

For a multi-inputs multi-outputs system, its output responses can be represented as follows,
\[ y = [y^1, y^2, \ldots, y^m] = [f^1(x), f^2(x), \ldots, f^m(x)]. \]  

(9)

First, a set of summatory function is constructed using any two output responses as follows,
\[
\begin{aligned}
G^{k,l}(x) &= f^k(x) + f^l(x) \\
H^{k,l}(x) &= f^k(x) - f^l(x), \quad 1 \leq k, l \leq m.
\end{aligned}
\]  

(10)

According to the high dimensional model representation in Eq. (1), the Eq. (9) can be rewritten by
\[
\begin{aligned}
G^{k,l}(x) &= G^{k,l}_0 + \sum_{i=1}^{n} G^{k,l}_i \left( x_i \right) + \sum_{1 \leq i < j \leq n} G^{k,l}_{ij} \left( x_i, x_j \right) + \cdots + G^{k,l}_{12..n} \left( x_1, x_2, \ldots, x_n \right) \\
H^{k,l}(x) &= H^{k,l}_0 + \sum_{i=1}^{n} H^{k,l}_i \left( x_i \right) + \sum_{1 \leq i < j \leq n} H^{k,l}_{ij} \left( x_i, x_j \right) + \cdots + H^{k,l}_{12..n} \left( x_1, x_2, \ldots, x_n \right)
\end{aligned}
\]  

(11)

The function sub-terms of Eq. (10) can be calculated referring to the Eqs. (3)~(5). According to Eq. (9), its response functions can be respectively represented by \( G^{k,l}(x) \) and \( H^{k,l}(x) \) as follows,
\[
\begin{aligned}
f^k(x) &= \frac{G^{k,l}(x) + H^{k,l}(x)}{2} \\
f^l(x) &= \frac{G^{k,l}(x) - H^{k,l}(x)}{2}
\end{aligned}
\]  

(12)

The sum of the variances of \( f^k(x) \) and \( f^l(x) \) can be calculated by
\[
\text{Va}(f^k(x)) + \text{Va}(f^l(x)) = \frac{1}{2} \left[ \text{Va}(G^{k,l}(x)) + \text{Va}(H^{k,l}(x)) \right]
\]  

(13)

where \( \text{Va}(\cdot) \) is a variance operator. In Eq. (12), \( \text{Va}(G^{k,l}(x)) \) and \( \text{Va}(H^{k,l}(x)) \) can be represented by
\[
\begin{aligned}
\text{Va}(G^{k,l}(x)) &= \text{Va} \left[ G^{k,l}_0 + \sum_{i=1}^{n} G^{k,l}_i \left( x_i \right) + \sum_{1 \leq i < j \leq n} G^{k,l}_{ij} \left( x_i, x_j \right) + \cdots + G^{k,l}_{12..n} \left( x_1, x_2, \ldots, x_n \right) \right] \\
&= DG^{k,l}_0 + \sum_{i=1}^{n} DG^{k,l}_i + \sum_{1 \leq i < j \leq n} DG^{k,l}_{ij} + \cdots + DG^{k,l}_{12..n}
\end{aligned}
\]  

(14)

The covariances of \( f^k(x) \) and \( f^l(x) \) can also be calculated as follows,
\[
\text{Cov}(f^k(x), f^l(x)) = \frac{1}{4} \left[ \text{Va}(G^{k,l}(x)) - \text{Va}(H^{k,l}(x)) \right]
\]  

(15)

where \( \text{Cov}(\cdot) \) is a covariance operator.

The total fluctuation of \( f^k(x) \) and \( f^l(x) \) can be calculated by the sum of variances and covariance, namely
\[
TF^{k,l} = \text{Va}(f^k(x)) + \text{Va}(f^l(x)) + \text{Cov}(f^k(x), f^l(x))
\]  

(16)
Substituting Eqs. (12)~(14) into Eq. (15), it can obtain the total fluctuation as follows,
\[
TF^{k,l} = \frac{3}{4}D\left(G_{0}^{k,l}\right) + \frac{1}{4}D\left(H_{0}^{k,l}\right) + \sum_{i=1}^{n}\left[\frac{3}{4}D\left(G_{i}^{k,l}(x_{i})\right) + \frac{1}{4}D\left(H_{i}^{k,l}(x_{i})\right)\right]
\]
\[+ \sum_{1 \leq i < j \leq n} \frac{3}{4}D\left(G_{ij}^{k,l}(x_{i},x_{j})\right) + \frac{1}{4}D\left(H_{ij}^{k,l}(x_{i},x_{j})\right)\]
\[+ \cdots + \frac{3}{4}D\left(G_{ij}^{k,l}(x_{1},x_{2},\cdots,x_{n})\right) + \frac{1}{4}D\left(H_{ij}^{k,l}(x_{1},x_{2},\cdots,x_{n})\right)\]
\[= TD_{0}^{k,l} + \sum_{i=1}^{n}TD_{i}^{k,l} + \sum_{1 \leq i < j \leq n} TD_{ij}^{k,l} + \cdots + TD_{12\cdots n}^{k,l}\]

where
\[
TD_{0}^{k,l} = \frac{3}{4}\text{Va}\left(G_{0}^{k,l}\right) + \frac{1}{4}\text{Va}\left(H_{0}^{k,l}\right)
\]
\[
TD_{i}^{k,l} = \frac{3}{4}\text{Va}\left(G_{i}^{k,l}(x_{i})\right) + \frac{1}{4}\text{Va}\left(H_{i}^{k,l}(x_{i})\right)
\]
\[
TD_{ij}^{k,l} = \frac{3}{4}\text{Va}\left(G_{ij}^{k,l}(x_{i},x_{j})\right) + \frac{1}{4}\text{Va}\left(H_{ij}^{k,l}(x_{i},x_{j})\right)
\]

For a MIMOS with \( n \) input variables and \( m \) output responses, a more general decomposition can be given as follows,
\[
TF^{1,2\cdots m} = \frac{3}{4}D\left(G_{0}^{1,2\cdots m}\right) + \frac{1}{4}D\left(H_{0}^{1,2\cdots m}\right) + \sum_{i=1}^{n}\left[\frac{3}{4}D\left(G_{i}^{1,2\cdots m}(x_{i})\right) + \frac{1}{4}D\left(H_{i}^{1,2\cdots m}(x_{i})\right)\right]
\]
\[+ \sum_{1 \leq i < j \leq n} \frac{3}{4}D\left(G_{ij}^{1,2\cdots m}(x_{i},x_{j})\right) + \frac{1}{4}D\left(H_{ij}^{1,2\cdots m}(x_{i},x_{j})\right)\]
\[+ \cdots + \frac{3}{4}D\left(G_{ij}^{1,2\cdots m}(x_{1},x_{2},\cdots,x_{n})\right) + \frac{1}{4}D\left(H_{ij}^{1,2\cdots m}(x_{1},x_{2},\cdots,x_{n})\right)\]
\[= TD_{0}^{1,2\cdots m} + \sum_{i=1}^{n}TD_{i}^{1,2\cdots m} + \sum_{1 \leq i < j \leq n} TD_{ij}^{1,2\cdots m} + \cdots + TD_{12\cdots m}^{1,2\cdots m}, \text{where } TD_{0}^{1,2\cdots m} = 0.\]

The sensitivity index for each decomposed function sub-term can be defined by
\[
FS_{i} = \frac{TD_{i}^{1,2\cdots m}}{TF^{1,2\cdots m}}, 1 \leq i_{1} < i_{2} < \cdots < i_{s} \leq n\]

The sensitivity index of variable is given by
\[
VS_{i}^{\text{tot}} = 1 - FS_{i},\]

### 3 Numerical Example and Engineering Application

To demonstrate the feasibility and practicability of the proposed GSA method, a numerical example and an engineering application are considered.

#### 3.1 Numerical Example

A set with two functions is constructed as follows,
\[
\begin{align*}
  f^1(x) &= x_1^2 + x_2 + x_1x_2^2, \quad x \in [0,1], \\
  f^2(x) &= x_1 - x_2^2 - x_1^2x_2
\end{align*}
\]

According to the variance decomposition, the partial variance matrix of \( f^1(x) \) and \( f^2(x) \) is \[
\begin{bmatrix}
0.154 & 0.189 & 0.007 \\
0.154 & 0.189 & 0.007 \\
0.022 & 0.154 & 0.007
\end{bmatrix}
\] . According to the covariance decomposition, the partial covariance vector of \( f^1(x) \) and \( f^2(x) \) is \[
\begin{bmatrix}
0.053 & -0.169 & -0.007 \\
0.053 & -0.169 & -0.007 \\
-0.022 & 0.154 & -0.007
\end{bmatrix}
\]. The SA results based on the average method and trace of covariance matrix are \([0.311,0.720] \) and \([0.358,0.670] \), respectively. Obviously, the results of these two methods ignore the influence of covariance. The SA results using the proposed method is \([0.577,0.442] \), which is different from the results of the average method and trace of covariance matrix due to taking into account the contribution of covariance. The comparisons of SA results are provided, as shown in Fig. 1. The numerical example demonstrates that the proposed method is more suitable for MIMOS than some traditional methods.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{Comparison of SA results using different methods.}
\end{figure}

3.2 Engineering Application

Nowadays, vehicle lightweight and safety design becomes an increasingly critical issue. Liu et al. [3] presented a composite B-pillar structure with ply drop-off to improve the crashworthiness of side impact and roof crush. The finite element model of the ply drop-off B-pillar consists of four parts, as shown in Fig. 2. The thickness of each part which depends on the number of lay-up is regarded as design variable. The design objectives consist of roof strength which stands for the maximum crushing force in roof crush, B-intrusion which stands for the intrusion of B-pillar in side impact, and B-velocity which stands for the maximum intrusion velocity of B-pillar in side impact. According to the samplings in Table 3 from Ref. [3], the response surface models on roof strength, B-velocity, and B-intrusion can be established as follows,
\[
\begin{align*}
RS(\mathbf{x}) &= -0.005x_1^2 - 0.001x_1x_3 - 0.001x_1x_4 + 0.363x_1 - 0.002x_2^2 - 0.001x_2x_4 \\
&+ 0.080x_2 - 0.006x_3^2 + 0.001x_3x_4 + 0.428x_3 + 0.068x_4 + 25.2 \\
BV(\mathbf{x}) &= 0.003x_1^2 - 0.128x_1 + 0.001x_1x_3 - 0.002x_1x_4 + 0.044x_2 \\
&- 0.055x_2 + 0.077x_2 + 12.5 \\
BI(\mathbf{x}) &= 0.089x_2^2 - 0.103x_2x_3 + 0.001x_3x_3 - 0.017x_2x_4 - 2.750x_3 + 0.079x_3^2 \\
&- 0.060x_2x_3 + 0.006x_2x_4 - 1.02x_3 + 0.075x_3^2 - 0.008x_3x_4 - 3.03x_3 \\
&+ 0.021x_4^2 - 0.557x_4 + 428.0
\end{align*}
\]

where \( \mathbf{x} \in [8,30] \) \( (23) \)

**Fig. 2** Structure design of composite B-pillar with ply drop-off [3].

Before the optimization design for the B-pillar with ply drop off, SA is implemented to assess the importance of variables. The SA results using the proposed method are 0.423, 0.058, 0.502, 0.016, respectively. The results indicate that the influence of variables \( x_1 \) and \( x_3 \) are important and the influence of variables \( x_2 \) and \( x_4 \) are ignored. It is helpful to implement further optimization design of B-pillar with ply drop-off.

**4 Conclusions**

Some traditional sensitivity analysis (SA) methods only consider the influence of variance decomposition and ignore the influence of covariance decomposition, which makes the SA results for MIMOS inaccurate. Hence, this paper develops a novel global sensitivity analysis method (GSA) based on variance and covariance decomposition to identify the importance of multi-inputs to multi-outputs. It is convenient to the variance and covariance decomposition of multi-output responses using a set of the constructed summatory functions. The function sensitivity index is defined by the ratio of the partial fluctuation of the decomposed function sub-term and total fluctuation. Further, the variable sensitivity index is calculated by the sum of all function sensitivity index including the effect of variable. The numerical example and engineering application demonstrate that the presented GSA method is feasible and practical.
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