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The robustness of Timoshenko beam in geometrically non-linear frame analysis

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Abstract

Tangent Stiffness Method using strict geometrical stiffness gives perfect equilibrium solutions with convergence of unbalanced forces at all nodes. In case of deep beam or dense mesh division, Timoshenko beam elements show better convergence against to the huge load which causes extremely large displacement than Euler-Bernoulli beam elements. In this study, a numerical example of large displacement analyses for 3-D frame structure with finite rotation is shown, and the performance of Timoshenko beam elements is discussed when the flat rectangular cross section is applied.

1 Introduction

The tangent stiffness method (TSM), which defines the element behavior in the element coordinate, gives a useful algorithm for large deformational analysis of 2-D and 3-D frame structures. The superiority of this method is that strict equilibrium solutions can be obtained by convergence of unbalanced solution using strict geometrical compatibility. We applied this method to 3-D analysis considering the finite rotations and obtained an equilibrium path with extremely large displacement (H.Obiya, K.Ijima, N.Kawasaki**,** 2000; K.Abe, H.Obiya, K.Ijima, 2007).

However, we defined the element force equation, which is a stiffness equation between the element edge forces and the element edge deformation, as linear based on Euler-Bernoulli beam. Therefore, in particular cases such as follows, we had problem of divergence of unbalanced forces when some elements have large deformations.

1) Cross section has extremely thin shape.
2) Number of mesh division exceeds a limit.

For example, in the case of the simulation of folding a ring into a third size, the cross section should be thin to observe stable behavior, but the problem of 1) was a bottleneck. Furthermore, the linear element force equations would require dense mesh division for realization of accuracy, against to the problem of 2).

In this study, we try two modifications of the element force equations as follows.

1) To the direction around the strong axis, Timoshenko beam theory that consider shear deformation would be applied. This modification would ensure the stable convergence, even if in case that the element length becomes short and/or that cross section becomes deep.

2) To the direction around the weak axis, the non-linear theory that consider the movement of the distance between both ends caused by bending.

This modification can give more accurate solutions than the linear element force equation, so we can expect the reduction of the number of mesh division. The numerical example in this paper shows the application of above two modifications can contribute the expansion of the coverage of TSM, thus a strict and robust algorithm for geometrical non-linear analysis can be realized.

2 Element edge force

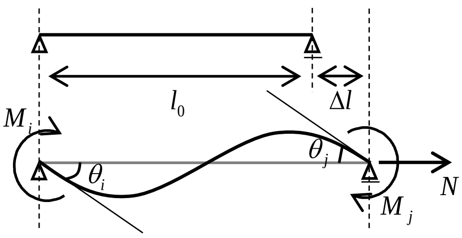


Fig.1 Element edge force and beam deformation in small deformation theory

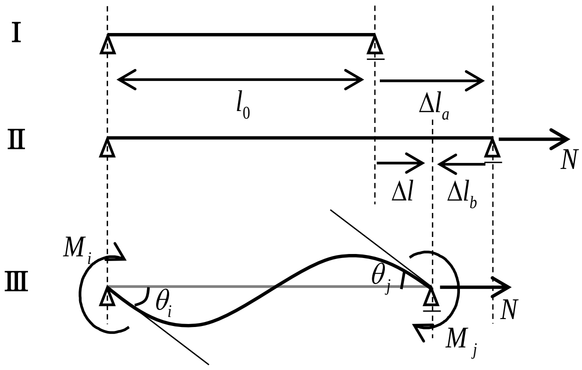


Fig.2 Element force edge and beam deformation in large deformation theory

2.1 *Euler-Bernoulli beam and Timoshenko beam theory in small deformation*

Let an element constituted by a stable and statically determinate support condition as shown in Fig.1. Element edge forces consist of axial force N, edge moments M_i and M_j are independent to each other. The element force equation for Euler-Bernoulli is shown in Eq. (1) and Timoshenko beam is shown in Eq. (2) could be expressed as a linear equation.

$$\begin{bmatrix} N \\ M_i \\ M_j \end{bmatrix} = \begin{bmatrix} F_0 & 0 & 0 \\ 0 & 4k & 2k \\ 0 & 2k & 4k \end{bmatrix} \begin{bmatrix} \Delta l \\ \theta_i \\ \theta_j \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} N \\ M_i \\ M_j \end{bmatrix} = \begin{bmatrix} F_0 & 0 & 0 \\ 0 & \frac{k}{(1+q)}(4+q) & \frac{k}{(1+q)}(2-q) \\ 0 & \frac{k}{(1+q)}(2-q) & \frac{k}{(1+q)}(4+q) \end{bmatrix} \begin{bmatrix} \Delta l \\ \theta_i \\ \theta_j \end{bmatrix} \quad (2)$$

$$F_0 = \frac{EA}{l_0},\ k = \frac{EI}{l_0},\ q = \frac{12EI}{GAl_0^2} \quad (3), (4), (5)$$

2.2 *Nonlinear Element force equation considering movement of string length*

When setting up the element forces and the element deformations in a line element whose area of cross section and moment of inertia are A and I, respectively, the bending moments at the edges are written as

$$\begin{bmatrix} M_i \\ M_j \end{bmatrix} = k \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} \tag{6}$$

$N > 0$

$$a = \frac{\omega^2 \cosh\omega - \omega \sinh\omega}{\omega \sinh\omega + 2(1-\cosh\omega)} \quad , \quad b = \frac{\omega \sinh\omega - \omega^2}{\omega \sinh\omega + 2(1-\cosh\omega)} \qquad (7), (8)$$

$N < 0$

$$a = \frac{\omega^2 \cos\omega - \omega \sin\omega}{\omega \sin\omega - 2(1-\cos\omega)} \quad , \quad b = \frac{\omega \sin\omega - \omega^2}{\omega \sin\omega - 2(1-\cos\omega)} \qquad (9), (10)$$

$$\omega = l_0 \sqrt{\frac{N}{EI}} \tag{11}$$

in which, E is Young's modules. Rewriting Eq. (6) into simple form, its differential calculus is the following.

$$\mathbf{M} = \mathbf{B}\boldsymbol{\theta} \tag{12}$$

$$\delta\mathbf{M} = \mathbf{B}\delta\boldsymbol{\theta} + \delta\mathbf{B}\boldsymbol{\theta} = \mathbf{B}\delta\boldsymbol{\theta} + \frac{d\mathbf{B}}{dN}\boldsymbol{\theta}\delta N = \mathbf{B}\delta\boldsymbol{\theta} + u\delta N \tag{13}$$

The axial force is proportionate to the difference between the curve length of the element and the nonstressed length.

$$N = \frac{EA}{l_0}(\Delta l + \Delta l_b) \tag{14}$$

in which Δl_b is the movement of string length which is distance between both element ends, caused by bending.

$$\Delta l_b = \frac{1}{2}\boldsymbol{\theta}^T \frac{d\mathbf{B}}{dN}\boldsymbol{\theta} = \frac{l_0}{4}\left\{p(\theta_i^2 + \theta_j^2) + 2\bar{p}\theta_i\theta_j\right\} \tag{15}$$

Therefore, the axial force depends on the bending deformations, so Eq. (13) becomes

$$\begin{bmatrix} \delta N \\ \delta M_i \\ \delta M_j \end{bmatrix} = \begin{bmatrix} F & Fu_1 & Fu_2 \\ Fu_1 & Fu_1^2 + ak & Fu_1u_2 + bk \\ Fu_2 & Fu_1u_2 + bk & Fu_2^2 + ak \end{bmatrix} \tag{16}$$

in which;

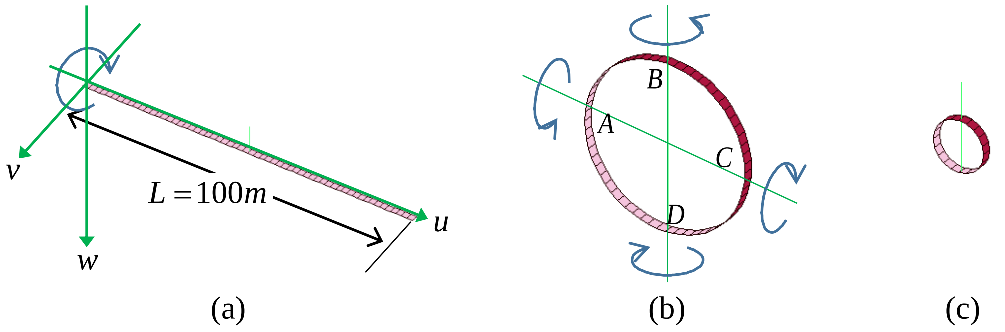
$$u_1 = \frac{l_0}{2}(p\theta_i - \bar{p}\theta_j) \ , \ u_2 = \frac{l_0}{2}(p\theta_j - \bar{p}\theta_i) \qquad (17), (18)$$

$$p = \frac{a - b^2}{\omega^2} \ , \ \bar{p} = \frac{ab - a - 2b}{\omega^2} \qquad (19), (20)$$

$$F = \frac{F_0}{1 + WF_0} \ , \ W = \frac{l_0^2}{4k}\left\{s(\theta_i^2 + \theta_j^2) - 2\bar{s}\theta_i\theta_j\right\} \qquad (21), (22)$$

$$s = \frac{1}{2\omega^4}\left\{a + 3b^2 - 2ab(b-1)\right\} \ , \ \bar{s} = \frac{1}{2\omega^4}\left[\left\{(a+b)^2 - a\right\}(b-1) - 2ab^2\right] \qquad (23), (24)$$

3 Numerical Example



$E = 210GPa$, $G = 81GPa$, Cross section: 1m×0.05m of rectangle

Fig.3 Initial form and analytical procedure

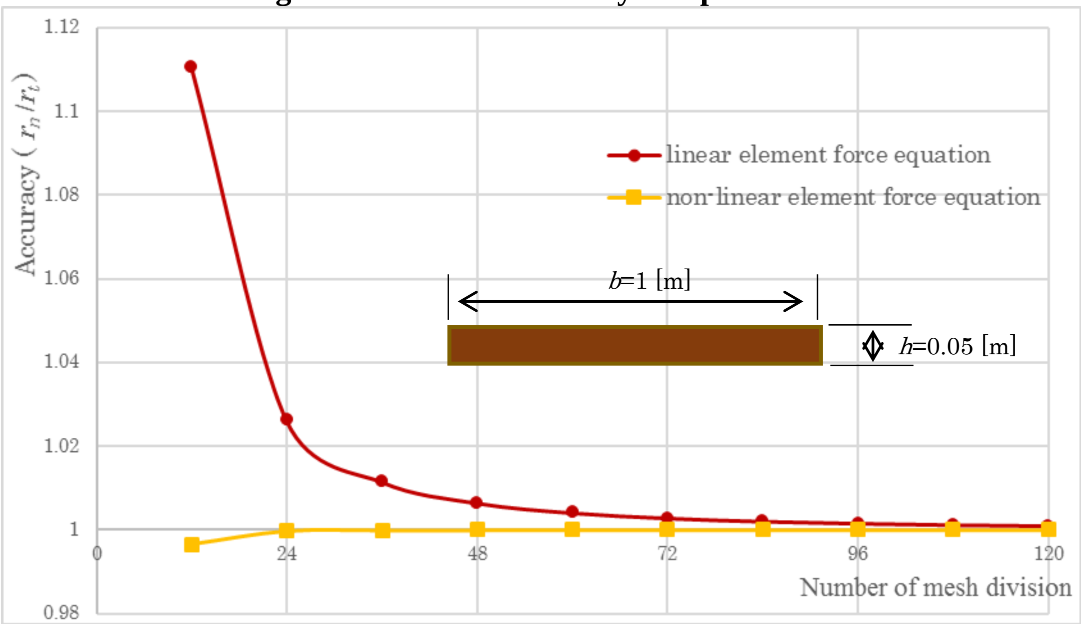
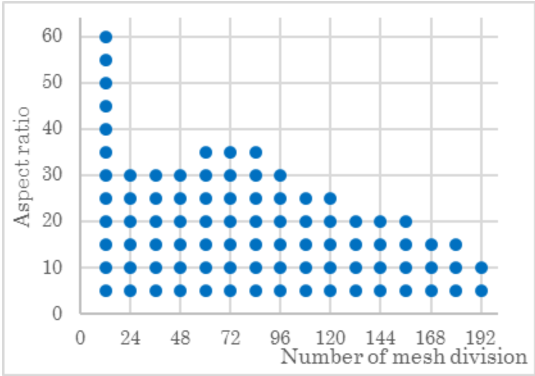
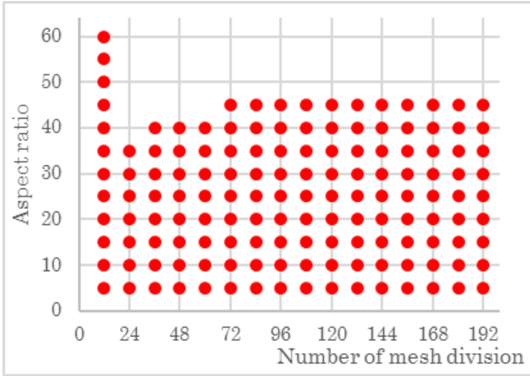


Fig.4 Accuracy of numerical solutions (Section aspect ratio:20)

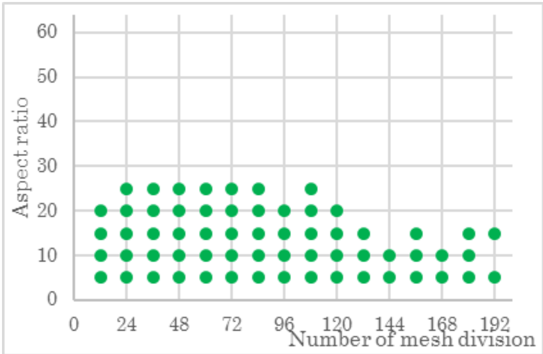
Fig.3 (a) shows the primary configuration of a cantilever beam, while Fig.3 (b) and (c) are the deformed sketch. In Fig.3 (a), compulsory rotational displacement around the v-direction is applied on the free edge of the cantilever beam. When 2π of rotational displacement is applied, the beam was deformed as Fig.3 (b) which results as a perfect circular shape. Further, when π of rotational displacement is applied on node of A-D to directions illustrated in Fig.3 (b), the beam deforms as Fig.3 (c), which exhibits a 3-layer circular shape with a third radius of primary. Fig.4 shows the ratio of r_n/r_t in which r_n is radius of numerical solutions and r_t is theoretical radius ($r_t = L / 6\pi$), in case of linear (Eq.(1), (2)) and nonlinear element force equations (Eq.(16)). Section aspect ratio in this paper is b/h shown in Fig.4. When using Eq. (16) as the element force equation, highly accurate result was obtained with small number of mesh division. Further, the error of accuracy is almost equal to 0.1 percent with 24 mesh when applying nonlinear element force equation to the direction around weak axis. On the other hand, if the linear element force equation to the direction around weak axis is applied, it is unable to satisfy 0.1 percent of error even with 120 meth. Therefore, it is clear that nonlinear element force equation of Eq. (16) ensures high accuracy with less mesh division and significantly more efficient than the linear element force equation of Eq. (1).



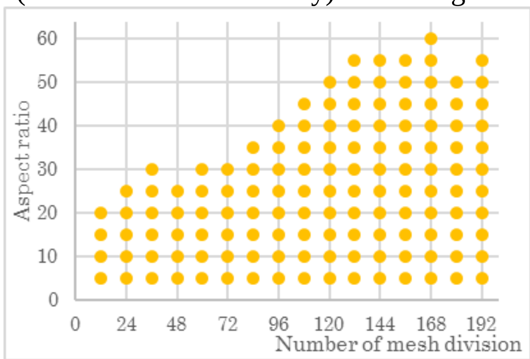
(a) Eq.(1) is applied to both axes



(b) Eq.(1) is to weak axis and Eq.(2) (Timoshenko beam theory)is to strong axis



(c) Eq.(1) is to strong axis and Eq.(16)(Nonlinear theory) is to weak axis



(d) Eq.(2) is to strong axis and Eq.(16) is to weak axis

Fig.5 Section aspect ratio and number of mesh division when a folding solution was obtained

In Fig.5, in order to evaluate the performance of the element force equations defined in this study, we examined the influence of section aspect ratio and number of mesh division to convergence of unbalanced forces. Every dot is indicated when the "folded solution" in Fig.3 (c) had been obtained corresponding to the conditions of the section aspect ratio and the number of mesh division. Namely, the conditions where no dot is indicated could not achieve convergence. If Euler-Bernoulli beam theory (Eq. (1)) is applied to the direction around strong axis (Fig.5 (a), (c)), thin and flat cross section is not available especially in case of dense mesh division. Fig.5 (b) shows that application of Timoshenko beam theory (Eq. (2)) to the direction around strong axis is effective to thin and flat cross section. On the other hand, when the nonlinear element force equation considering the movement of string length (Eq. (16)) is applied, even rough mesh division can provide strict solutions, but thin and flat cross section seems to disturb convergence.

4 Conclusion

Based on the findings of this study, application of Timoshenko beam theory is effective to avoid divergence of unbalanced force in case of dense mesh division and/or thin rectangular cross section. Furthermore, to ensure highly accurate result when dealing with a large displacement analysis which consumes huge calculation cost, we can use the nonlinear element force equation considering the movement of string length on the weak axis. Consequently, the combination of the Timoshenko beam theory and the nonlinear element force equation considering the movement of string length supplement the versatility and robustness of TSM.

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