A new assumed strain beam element based on a sixth-order beam theory for static and dynamic analysis of composite beams

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Abstract

An accurate, efficient and reliable two-noded beam element is presented in this paper for the static and dynamic analysis of laminated composite beams. The element formulation is based on the quasi-conforming element technique and a new sixth-order beam theory proposed by the second author in which the average rotation of beam cross-section is employed as the independent field variable instead of the rotation at the beam midplane used in other higher-order beam theories. The element stiffness matrix of the resulting beam element is given explicitly; consequently it is very computationally efficient. Furthermore, this new shear flexible beam element exhibits higher accuracy than the conventional shear flexible beam elements, as it possesses a linear bending strain field although there are only three nodal parameters associated with bending deformation at each node. Both static and dynamic analyses of laminated composite beams with different aspect ratios and boundary conditions are solved. The numerical results clearly demonstrate that the present composite beam element is not only efficient and locking free but also very accurate. The free vibration analysis of composite beams also indicates that the use of the average rotation of the beam cross-section improves the prediction accuracy of the higher-mode flexural frequencies.

Keywords: Shear flexible beam element, Sixth-order beam theory, Quasi-conforming element technique, Composite beam, Higher-mode flexural vibration

Introduction

Laminated composite beams are widely used in various engineering structures because of the high specific stiffness and high strength. The transverse shear deformations play an important role in the analysis of composite beams. Various shear deformable beam theories and shear flexible beam elements have been proposed for the static and dynamic analyses of composite beams in the past few decades (Kapania and Raciti, 1989; Chandrashekhara and Bangera, 1993; Shi *et al*., 1998, 1999; among others). The theoretical and numerical modeling of composite beams is still attracting many researchers' attention even today (Carrera and Giunta, 2010; Feng *et al.*, 2012). Among all the refined beam theories, the simple third-order shear deformation beam theory presented by Bickford (1982) or reduced from Reddy plate theory (1984) is very attractive in the finite element modeling of composite beams, as it does not need the shear correction factors and the warping of the cross-section can be accounted for to a certain extent.

Shi and Voyiadjis (1991) demonstrated that the assumed strained method what is based on the quasi-conforming element technique (Tang *et al.*, 1980) is a very efficient approach to formulate the shear flexible arch/beam elements and shell elements, since the resulting elements are not only free from shear locking, but also free from the time consuming numerical integration. Shi *et al*. (1998, 1999) presented efficient and accurate two-noded composite beam elements based on the third-order shear deformation beam theory. The composite beam elements developed by Shi *et al*. (1998, 1999) yield accurate results for the static analysis and the lower-mode frequencies of flexural vibration of composite beams. However, the accuracy of the predicted higher-mode frequencies, e.g. the fourth and fifth-mode frequencies, is not good enough (Shi and Lam, 1999).

Hutchinson (1986) studied the influence of the different definition of the rotation variable used in plate theories on the accuracy of higher-mode frequencies of the clamped plates. Hutchinson showed that the plate theory in which average rotation across the plate thickness is used can correctly predict both the fundamental natural frequency and the higher-mode frequencies of the clamped plates, but the theory in which the midplane rotation is used can only give the correct solutions of the first and second natural frequencies, but predicts physically impossible results for the third and fourth natural frequencies. Shi and Voyiadjis (2011) proposed a beam theory with the sixth-order differential equations (for bending only) based on a refined third-order transverse shear function which is similar to that used in Bickford beam theory. However, one major difference of this new beam theory from other higher-order beam theories is that the averaged rotation of the beam cross-section is defined as a displacement variable as opposed to the rotation measured at the beam midplane used in other higher-order beam theories. Wang and Shi (2012) demonstrated that this new sixth-order beam theory is not only accurate but also capable of predicting correct boundary layer solutions at the locations with displacement boundary conditions.

The objective of this paper is to present a new composite beam element with the averaged rotation of beam cross-section as one of the nodal degrees of freedom by using the sixth-order beam theory proposed by Shi and Voyiadjis (2011) and the quasi-conforming element technique**.** The resulting two-noded beam element is not only free from the shear locking, but also free from the numerical integration. Both static and dynamic analyses of composite beams with various aspect ratios and boundary conditions are solved here to evaluate the accuracy and reliability of the new composite beam element. The results of numerical examples show that the sixth-order beam theory proposed by Shi and Voyiadjis (2011) can yield more accurate results, especially the higher-mode flexural frequencies of composite beams than Bickford beam theory.

The static analysis of finite element formulation of composite beams based on the sixth-order beam theory

The new finite element formulation of composite beams is based on the sixth-order beam theory proposed by Shi and Voyiadjis (2011), which has been proved a high efficiency and accuracy sixth-order theory in both static and dynamic analysis.

Displacement fields and strains of shear deformable beams

The displacement field in the sixth-order beam theory proposed by Shi and Voyiadjis (2011) is of the form

$$u(x,z,t) = u_0(x,t) - z(\frac{\partial w_0}{\partial x} - \gamma) + (\alpha z - \beta z^3)\gamma\ ,\ \ w(x,z,t) = w_0(x,t) \tag{1}$$

where u_0 and w_0 are the axial displacement and the deflection of a point on the beam reference plane respectively; $\overline{\phi}_x$ is the averaged rotation of the beam cross-section through the beam thickness; γ is the transverse shear strain of the beam cross-section; h is the beam thickness; $\alpha = 1/4$ and $\alpha = 5/(3h^2)$. The transverse shear strain γ takes the form

$$\gamma = \frac{\partial w_0}{\partial x} + \overline{\phi}_x \tag{2}$$

It follows from Eq. (1) that the normal strain and the transverse shear strain under consideration take the form

$$e_x = e_m - z e_b + (\alpha z - \beta z^3) e_{hs}\,,\; 2e_{xz} = (\frac{5}{4} - \frac{5z^2}{h^2}) e_s \tag{3}$$

with

$$e_m = \frac{\partial u_0}{\partial x}\,,\; e_b = \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial \gamma}{\partial x}\,,\; e_{hs} = \frac{\partial \gamma}{\partial x}\,,\; e_s = \gamma \tag{4}$$

The expression of strains above results in a C^1-continuity element under the displacement-based formulation. The strains in Eq. (4) are the functions of the deflection and transverse shear deformation. Corresponding to the strains defined in Eq. (4), the simplest nodal degrees of freedom at node i, $\boldsymbol{q}_i$ can be chosen as

$$\mathbf{q_i} = [u_{0i}, w_{0i}, (\frac{\partial w_0}{\partial x})_i, \gamma_i]^T\,,\; i = 1, 2 \tag{5}$$

The nodal variables in Eq. (5) result in a cubic approximation for deflection w_0 and a linear transverse shear strain γ. Then Eq. (4) can give a linear element bending strain. Because the bending strain is the dominant term in bending problems, then in finite element analysis, the strain expressions derived from the displacement defined in Eq. (1) should lead to a more accurate solution than those higher-order beam theories which give a constant bending strain over an element, even though they have the same number of degrees of freedom at each node (Shi *et al.*, 1998).

Stiffness matrix of the sixth-order composite beam element

Now consider a straight beam of length *l* and rectangular cross-section with thickness *h* and width *b*. The strain energy density of the beam, *U* is of the form

$$U_e = \frac{b}{2}\int_l \int_{-h/2}^{h/2} (e_x Q_{xx} e_x + 4 e_{xz} Q_{xz} e_{xz}) dz dx \tag{6}$$

where Q_{xx} and Q_{xz} are the longitudinal Young's modulus and transverse shear modulus respectively, and they are functions of *z*. Substituting Eqs. (3, 4) into Eq. (6) leads to

$$\begin{aligned} U_e = \frac{1}{2}\int_l [& e_m A_{xx} e_m + e_b D_{xx} e_b + e_{hs}(\alpha^2 D_{xx} - 2\alpha\beta F_{xx} + \beta^2 H_{xx}) e_{hs} + \gamma S_{xx}\gamma \\ & - B_{xx}(e_b e_m + e_m e_b) + (\alpha B_{xx} - \beta E_{xx})(e_m e_{hs} + e_{hs} e_m) + (\beta F_{xx} - \alpha D_{xx})(e_b e_{hs} + e_{hs} e_b)] dx \end{aligned} \tag{7}$$

in which

$$A_{xx}, B_{xx}, D_{xx}, E_{xx}, F_{xx}, H_{xx} = b\int_{-h/2}^{h/2} (1, z, z^2, z^3, z^4, z^6) Q_{xx} dz\,,\; S_{xx} = b\int_{-h/2}^{h/2} (\frac{5}{4} - \frac{5z^2}{h^2})^2 Q_{xz} dz \tag{8}$$

The element strains in Eq. (7) can be expressed in terms of the element nodal displacement $\mathbf{q}$ and the element strain matrices as follows

$$e_m = \mathbf{B_m}\mathbf{q}_e\,,\; e_b = \mathbf{B_b}\mathbf{q_e}\,,\; e_{hs} = \mathbf{B_{hs}}\mathbf{q_e}\,,\; \gamma = \mathbf{B_s}\mathbf{q_e} \tag{9}$$

Consequently, the strain energy in an element of length *l*, $\prod_e$ takes the form

$$\begin{aligned} \prod_e = \frac{1}{2}\mathbf{q_e^T}\int_l [& \mathbf{B_m^T} A_{xx}\mathbf{B_m} + \mathbf{B_b^T} D_{xx}\mathbf{B_b} + \mathbf{B_{hs}^T}(\alpha^2 D_{xx} - 2\alpha\beta F_{xx} + \beta^2 H_{xx})\mathbf{B_{hs}} \\ & + \mathbf{B_s^T} S_{xx}\mathbf{B_s} - B_{xx}(\mathbf{B_m^T}\mathbf{B_b} + \mathbf{B_b^T}\mathbf{B_m}) + (\alpha B_{xx} - \beta E_{xx})(\mathbf{B_m^T}\mathbf{B_{hs}} + \mathbf{B_{hs}^T}\mathbf{B_m}) \\ & + (\beta F_{xx} - \alpha D_{xx})(\mathbf{B_b^T}\mathbf{B_{hs}} + \mathbf{B_{hs}^T}\mathbf{B_b})] dx \mathbf{q_e} \end{aligned} \tag{10}$$

If we define element bending, membrane, transverse shear, higher-order shear and coupling stiffness matrix, respectively, as

$$\mathbf{K_b} = \int_l \mathbf{B_b^T} D_{xx}\mathbf{B_b} dx\,,\; \mathbf{K_m} = \int_l \mathbf{B_m^T} A_{xx}\mathbf{B_m} dx\,,\; \mathbf{K_s} = \int_l \mathbf{B_s^T} S_{xx}\mathbf{B_s} dx \tag{11a}$$

$$\mathbf{K_{hs}} = \int_l \mathbf{B_{hs}^T} (\alpha^2 D_{xx} - 2\alpha\beta F_{xx} + \beta^2 H_{xx}) \mathbf{B_{hs}} dx \tag{11b}$$

$$\mathbf{K_c} = \int_l -B_{xx} (\mathbf{B_m^T B_b} + \mathbf{B_b^T B_m}) + (\alpha B_{xx} - \beta E_{xx})(\mathbf{B_m^T B_{hs}} + \mathbf{B_{hs}^T B_m}) + (\beta F_{xx} - \alpha D_{xx})(\mathbf{B_b^T B_{hs}} + \mathbf{B_{hs}^T B_b}) dx \tag{11c}$$

Then the element stiffness matrix **K** is of the form

$$\mathbf{K_e} = \mathbf{K_b} + \mathbf{K_m} + \mathbf{K_s} + \mathbf{K_{hs}} + \mathbf{K_c} \tag{12}$$

Element strain matrix obtained from the quasi-conforming element technique

In conventional displacement-based finite element formulation, the element strain matrices in Eq. (9) are obtained from the interpolated displacement function in the element. However, these matrices will be evaluated by the quasi-conforming element technique (Tang *et al.*, 1980) in this work. For a quasi-conforming element, the element strain field is interpolated directly over the element domain rather than differentiated from the assumed displacement field, and the compatibility in an element domain is satisfied in a weak form. Let a *prime* signify the assumed element strain field, then the element strain energy in Eq. (10) can be modified as

$$\Pi_e^* = \Pi_e + \int_l \tilde{M} (e_b - e_b')dx + \int_l \tilde{N}(e_m - e_m')dx + \int_l \tilde{Q}(e_s - e_s')dx + \int_l \tilde{P}(e_{hs} - e_{hs}')dx \tag{13}$$

where $\tilde{M}$, $\tilde{N}$, $\tilde{Q}$ and $\tilde{P}$ are the test functions corresponding to their relevant strains. A cubic displacement w_0, a linear transverse shear strain γ and a linear displacement u_0 can be interpolated over the element from the element nodal variables. Then a suitable element stain field for the strains defined in Eq. (4) can be approximated as

$$e_b = \frac{d^2 w_0}{dx^2} - \frac{d\gamma}{dx} \approx e_b' = \alpha_{b1} + x\alpha_{b2} \, , \; e_m = \frac{du_0}{dx} \approx e_m' = \alpha_m \, , \; 2e_s = \gamma \approx 2e_s' = \alpha_s \, , \; e_{hs} = \frac{d\gamma}{dx} \approx e_{hs}' = \alpha_{hs} \tag{14}$$

where α_{bi} *(i=1, 2)*, α_m , α_s and α_{hs} are the assumed element strain parameters which can be determined from the weak form of compatibility given in Eq. (13) at element level. A linear bending strain is assumed here which is corresponding to the cubic deflection w_0 given by the element nodal displacements. The assumed constant transverse shear strain in Eq. (14) is one order lower than the interpolation given by the two nodal shear strain variables, which is equivalent to the reduced integration in the displacement-based formulation. In the quasi-conforming element formulation, $2e_s$ and e_{hs} can be approximated independently. It should be pointed out that the bending strain defined in this way optimally utilizes the given nodal variables since the bending strain is the dominant term in the beam analysis. The shear locking can be avoided by the quasi-conforming element technique (Shi *et al.*, 1991, 1998). By substituting the matrices **B** into Eqs. (11) and (12), the element stiffness matrix can be obtained. Therefore, the resulting element stiffness matrix can be evaluated explicitly, i.e. no numerical integration is needed, which makes the resulting beam element very computationally efficient.

The dynamic analysis of finite element modeling of composite beams based on Shi's sixth-order beam theory

Velocities of shear deformable beams

It follows from the Eq. (1) that the velocities in the *x*-direction and *z*-direction respectively take the form

$$v_x = \frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} - \frac{\partial}{\partial t}(\frac{\partial w_0}{\partial x} - \gamma)z + (\alpha z - \beta z^3)\frac{\partial \gamma}{\partial t} \, , \; v_z = \frac{\partial w_0}{\partial t} \tag{15}$$

Equation of motion of composite beam element

In dynamic analysis, the equation of motion is derived in terms of the element stiffness matrix and the mass matrix from Hamilton's Principle. U_e and K_{ke} are the element strain energy and kinetic energy respectively, and then the Hamilton's Principle states that

$$\delta\sum_{elem}\int_{t_0}^{t}(U_e - K_{ke})dt = 0 \tag{16}$$

In dynamic analysis of natural frequency of system, the work done by external forces is neglected and the damping is not considered. And Eq. (16) leads to the equilibrium equations of a system as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{17}$$

where $\mathbf{M}$ and $\ddot{\mathbf{q}}$ are the global mass matrix and acceleration vector of the system. Consequently, the frequency ω can be evaluated by

$$(\mathbf{K}\text{-}\omega^2\mathbf{M})\mathbf{q} = \mathbf{0} \tag{18}$$

The mass matrix based on the sixth-order beam theory will be presented in next section.

Consistent mass matrix for sixth-order beam theory

The kinetic energy of an element K_{ke} corresponding to the sixth-order theory takes the form

$$\begin{aligned} K_{ke} &= \frac{b}{2}\int_l\int_{-h/2}^{h/2}(v_x^2 + v_z^2)\rho(z)dzdx = \frac{b}{2}\int_l\int_{-h/2}^{h/2}[(\frac{\partial w}{\partial t})^2 + (\frac{\partial u}{\partial t})^2]\rho(z)dzdx \\ &= \frac{b}{2}\int_l\int_{-h/2}^{h/2}[(\frac{\partial w_0}{\partial t})^2 + (\frac{\partial u_0}{\partial t})^2 + z^2(\frac{\partial^2 w_0}{\partial t\partial x})^2 + ((1+2\alpha+\alpha^2)z^2 - 2(\beta+\alpha\beta)z^4 + \beta^2 z^6)(\frac{\partial \gamma}{\partial t})^2 \\ &\quad - 2z\frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial t\partial x} + 2((\alpha+1)z - \beta z^3)\frac{\partial u_0}{\partial t}\frac{\partial \gamma}{\partial t} + 2((-1-\alpha)z^2 + \beta z^4)\frac{\partial \gamma}{\partial t}\frac{\partial^2 w_0}{\partial t\partial x}]\rho dzdx \end{aligned} \tag{19}$$

where $\rho(z)$ is the density across the beam thickness. By defining

$$J_A, J_B, J_D, J_E, J_F, J_H = b\int_{-h/2}^{h/2}(1, z, z^2, z^3, z^4, z^6)\rho dz \tag{20}$$

The element kinetic energy K_{ke} can be written as

$$\begin{aligned} K_{ke} &= \frac{1}{2}\int_l J_A(\frac{\partial w_0}{\partial t})^2 + J_A(\frac{\partial u_0}{\partial t})^2 + J_D(\frac{\partial^2 w_0}{\partial t\partial x})^2 + ((1+2\alpha+\alpha^2)J_D - 2(\beta+\alpha\beta)J_F + \beta^2 J_H)(\frac{\partial \gamma}{\partial t})^2 \\ &\quad -2J_B\frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial t\partial x} + 2((\alpha+1)J_B - \beta J_E)\frac{\partial u_0}{\partial t}\frac{\partial \gamma}{\partial t} + 2((-1-\alpha)J_D + \beta J_F)\frac{\partial \gamma}{\partial t}\frac{\partial^2 w_0}{\partial t\partial x}]dx \end{aligned} \tag{21}$$

The equation above shows that similar to the stretching and bending coupling in the stiffness matrix there is also an axial and rotary velocity coupling in the mass matrix when the density is not symmetric about the reference plane of the composite beams. The coupling of the transverse shear velocity and the deflection slope velocity is always non-zero as long as the transverse shear deformation is not zero. In the finite element analysis, the element displacement can be interpolated in terms of the element nodal displacement vector $\mathbf{q_e}$ as

$$u_0 = \mathbf{N_u}\mathbf{q_e},\ w_0 = \mathbf{N_w}\mathbf{q_e},\ \frac{\partial w_0}{\partial x} = \mathbf{N_{wx}}\mathbf{q_e},\ \gamma = \mathbf{N_\gamma}\mathbf{q_e} \tag{22}$$

where $\mathbf{N}_j$ (j=u, w and γ) are the interpolation matrices. By substituting Eq. (21) and (22) into (16), one obtains the consistent element mass matrix $\mathbf{M_e}$ as

$$\mathbf{M_e} = \mathbf{M_w} + \mathbf{M_u} + \mathbf{M_{wx}} + \mathbf{M_\gamma} + \mathbf{M_{uw}} + \mathbf{M_{u\gamma}} + \mathbf{M_{wx\gamma}} \tag{23}$$

with

$$\mathbf{M}_{\mathbf{w}} = \int_l \mathbf{N}_{\mathbf{w}}^{\mathbf{T}} J_A \mathbf{N}_{\mathbf{w}} dx\,,\ \mathbf{M}_{\mathbf{wx}} = \int_l \mathbf{N}_{\mathbf{wx}}^{\mathbf{T}} J_D \mathbf{N}_{\mathbf{wx}} dx\,,\ \mathbf{M}_{\mathbf{u}} = \int_l \mathbf{N}_{\mathbf{u}}^{\mathbf{T}} J_A \mathbf{N}_{\mathbf{u}} dx$$

$$\mathbf{M}_{\gamma} = \int_l \mathbf{N}_{\gamma}^{\mathbf{T}} [(1+\alpha)^2 J_D - 2\beta(1+\alpha) J_F + \beta^2 J_H] \mathbf{N}_{\gamma} dx\,,\ \mathbf{M}_{\mathbf{uw}} = -\int_l [\mathbf{N}_{\mathbf{u}}^{\mathbf{T}} J_B \mathbf{N}_{\mathbf{w}} + \mathbf{N}_{\mathbf{w}}^{\mathbf{T}} J_B \mathbf{N}_{\mathbf{u}}] dx$$

$$\mathbf{M}_{\mathbf{u}\gamma} = \int_l [\mathbf{N}_{\mathbf{u}}^{\mathbf{T}} ((\alpha+1) J_B - \beta J_E) \mathbf{N}_{\gamma} + \mathbf{N}_{\gamma}^{\mathbf{T}} ((\alpha+1) J_B - \beta J_E) \mathbf{N}_{\mathbf{u}}] dx \tag{24}$$

$$\mathbf{M}_{\mathbf{wx}\gamma} = \int_l [\mathbf{N}_{\mathbf{wx}}^{\mathbf{T}} ((-\alpha-1) J_D + \beta J_F) \mathbf{N}_{\gamma} + \mathbf{N}_{\gamma}^{\mathbf{T}} ((-\alpha-1) J_D + \beta J_F) \mathbf{N}_{\mathbf{wx}}] dx$$

$\mathbf{M}_{\mathbf{w}}$, $\mathbf{M}_{\mathbf{u0}}$ and $\mathbf{M}_{\mathbf{wx}}$ are, respectively, the usual transverse, axial and rotary inertia matrices; $\mathbf{M}_{\gamma}$ is the mass matrix resulting from the higher-order displacement; and $\mathbf{M}_{\mathbf{u0w}}$, $\mathbf{M}_{\mathbf{u0}\gamma}$ and $\mathbf{M}_{\mathbf{wx}\gamma}$ are the coupling terms of different components of the axial displacement. The variational consistent mass matrix above can account for the contribution of the higher-order displacement to the mass matrix and the results show that the consistent mass matrix can provide more accurate results than those given by lump mass matrix (Shi and Lam, 1999).

Numerical examples

The accuracy of new finite element formulation based on Shi and Voyiadjis' sixth-order beam theory is evaluated by three examples of statics and dynamics of composite beams with different aspect ratios boundary conditions in this section.

Example 1 simply supported composite beam under uniformed load

A four layered composite beam of rectangular cross-section is considered in this example. Its length to thickness ratio is 100, and its mechanical and geometrical properties are shown bellow.

$E_1 = 144.8GPa$, $E_2 = 9.65GPa$, $G_{12} = G_{13} = 4.14GPa$, $G_{23} = 3.45GPa$, $\nu_{12} = 0.3$, *L/h*=100, *h/b*=1

Table 1. Non-dimensional* deflections of simply supported composite beam

Lamination schemes	Element formulations	No. of elements		
		4	8	16
[0]	HQCB-8A (Shi *et al*, 1998)	0.08267	0.08597	0.08677
	Chandrashekhara *et al* (1993)	0.06282	0.07519	0.08155
	Present	0.08273	0.08598	0.08679
[0/90/90/0]	HQCB-8A (Shi *et al*, 1998)	0.09369	0.09738	0.09827
	Chandrashekhara *et al* (1993)	0.07107	0.08551	0.09225
	Present	0.09364	0.09733	0.09825

*the non-dimensional deflection is defined in Shi *et al.* (1998)

The beam element of HQCB-8A in the table is a C^1 composite beam element presented by Shi *et al.* (1998) which is based on Bickford beam theory (1982) and the quasi-conforming element technique. The results of Chandrashekhara *et al.* (1993) in Table 1 were given by a beam element based on the same type beam theory and the conventional displacement-based element formulation. It can be seen from the results in the table that the new beam element based on the sixth-order beam theory of Shi and Voyiadjis yield almost the same results as HQCB-8A in this numerical example; and the present beam element gives much better results than the conventional beam element.

Example 2 Laminated composite beams with different aspect ratios and different boundary conditions under the action of uniformed distributed load

Four equal thickness laminated composite beams with different boundary conditions and different length to height ratios are considered in this example. The material properties of the laminated composite beams are the same as the previous example. The lamination scheme of this example is [0/90/90/0]. The boundary conditions here include the clamped-clamped ends (denoted by CC); the clamped-free ends (CF); the clamped-simply supported ends (CS) and the simply supported ends

(SS). The non-dimensional maximum deflections of the beams given by various methods are listed in Table 2. Eight elements are used in the present analysis. The results in the table show that the beam element presented in this paper is not only free from shear locking but also very accurate.

Table 2 Influence of boundary conditions and aspect ratios on the accuracy of solutions

Boundary conditions	Element types	Aspect ratio *L/h*			Analytical Solution of thin beams (Timoshenko *et al*, 1972)
		15	100	1000	
CC	HQCB-8A (Shi *et al*, 1999)	0.03344	0.01998	0.01966	0.01967
	Kadoli *et al* (2008)	0.02993	-	-	
	Present	0.03344	0.01997	0.01965	
CF	HQCB-8A (Shi *et al*, 1999)	1.0038	0.9497	0.9481	0.9439
	Kadoli *et al* (2008)	0.9108	-	-	
	Present	1.0034	0.9503	0.9485	
CS	HQCB-8A (Shi *et al*, 1999)	0.05657	0.04113	0.04078	0.04090
	Kadoli *et al* (2008)	0.0517	-	-	
	Present	0.05600	0.04088	0.04052	
SS	HQCB-8A (Shi *et al*, 1999)	0.1112	0.09738	0.09706	0.09831
	Kadoli *et al* (2008)	0.1035	-	-	
	Present	0.1111	0.09733	0.09702	

Example 3 Laminated composite beam under uniformed load with different boundary conditions and different aspect ratios

The four layered laminated composite beams considered in Example 2 with the density of 1389.23 kg/m^3 are taken in this example for free vibration analysis. Twenty elements are used for the whole beam. The first five non-dimensional flexural frequencies (designated by *f(1)*-*f(5)*) of thick composite beams (*L/h=15*) are tabulated in Table 3. The nondimensional frequency is defined by $\overline{\omega}_i = \omega_i L^2 \sqrt{J_A/(E_1 b h^3)}$. Some other numerical results and analytical solutions are also listed in Table 3 for comparison. The numerical solutions of ABAQUS are obtained by the 8-noded solid elements, and two-layers of solid elements are used for each lamina of the laminated beam.

Table 3. Nondimensional frequencies of symmetric [0/90/90/0] cross-ply beams with *L/h =15*

Formulation	BCs	Nondimensional frequency at various vibration modes					
		f(1)	*f(2)*	*f(3)*	*a(1)*	*f(4)*	*f(5)*
Present (20 elements)	SS	2.4952	8.4551	15.7208	17.2113	23.2552	30.7395
	CC	4.6228	10.4438	17.2895	-	24.4919	31.7930
	CS	3.5313	9.4970	16.5073	17.2110	23.8354	31.1695
	CF	0.9163	4.9085	11.5193	17.2110	18.8428	26.3646
HQCB-8A (Shi *et al.* 1998)	SS	2.4979	8.4364	15.5932	-	22.8974	30.0061
	CC	4.6194	10.4162	17.1724	-	24.2001	31.2144
	CS	3.5264	9.4736	16.4201	-	23.5591	30.6107
	CF	0.9199	4.9054	11.4886	-	18.6886	25.9931
Chandrashekhara *et al.* (1993) Analytical solutions	SS	2.5023	8.4812	15.7558	-	23.3089	30.8386
	CC	4.5940	10.2906	16.9559	-	24.1410	31.2874
	CS	3.5254	9.4423	16.3839	-	23.6850	31.0569
	CF	0.9241	4.8925	11.4400	-	18.6972	26.2118
ABAQUS (80×8 mesh)	SS	2.4862	8.4415	15.7185	-	23.3170	30.9390
	CC	4.5866	10.3185	17.0647	-	24.2398	31.6197
	CS	3.5108	9.4336	16.4144	17.1934	23.7902	31.2835
	CF	0.9181	4.8749	11.4333	17.1920	18.7360	26.3304

The frequencies of the axial vibration, denoted by *a(1)*, of the beams with boundaries of SS, CS and CF predicted by the present beam element are also listed in the table. The *a(1)* of 3D composite beams are given by ABAQUS for the beams with the boundary $u(0, y, z) = 0$. One can see from the results in Table 3 that the present element which is based on the sixth-order beam theory of Shi and Voyiadjis (2011) yields more accurate results especially for the higher-mode frequencies than the beam element HQCB-8A which is based on the third-order shear beam theory of Bickford (1982). The major difference between the sixth-order beam theory of Shi and Voyiadjis and the third-order shear beam theory of Bickford lies on the different definitions of the rotation of the cross-section; the former employs an averaged rotation across the beam cross-section and the later uses the rotation measured at the beam midplane. As matter a fact, Hutchinson (1986) showed that the third-order shear plate theory yielded the incorrect results of the higher-mode frequencies of a clamped circular plate when the rotation along the clamped boundary was fixed at the plate midplane.

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Conclusions

This paper presents a new beam element which is based on the quasi-conforming element technique and a new beam theory with the averaged rotation of beam cross-section as one of the field variable proposed by Shi and Voyiadjis**.** There are two conclusions can be made from the result comparisons conducted in the paper.

1. The present assumed strain beam element is not only free from shear-locking as well as free from the time consuming numerical integration, but also very accurate.
2. The averaged rotation across the beam cross-section used in the sixth-order beam theory of Shi and Voyiadjis yield more accurate higher-mode frequencies than the high-order beam theories with the rotation of the cross-section measured at the beam midplane.

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