Interval method for solving the dynamics problems of multibody system with uncertain parameters

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The theoretical and computational aspects of interval methodology based on Chebyshev polynomials for modeling complex nonlinear multi-body dynamic systems in the presence of parametric and external excitation uncertainties is formulated, implemented, and validated. Both the parameters uncertainties and external excitation uncertainties are modeled by uncertain-but-bounded interval variables, where the bounds on the magnitude of uncertain parameters or external force are only required, not necessarily knowing the probabilistic distribution densities. The Chebyshev inclusion function which employs the truncated Chevbyshev series expansion to approximate the original nonlinear calculates sharper range than the traditional Taylor inclusion function. The coefficients of the Chebyshev polynomials are calculated through the Mehler numerical integral method. The multi-body systems dynamics are governed by differential algebraic equations (DAEs) which are transformed to nonlinear equations with interval parameters at each integral step by HHT-I3 methods, and then the proposed method for nonlinear systems with interval parameters can be employed to find the interval region of the system responses. The numerical example results show that the novel methodology can reduce the overestimation largely and is computationally faster than the scanning method.

Keywords: interval method; Chebyshev polynomials; uncertain analysis; DAEs

Introduction

Modern multibody systems containing such as mechanisms, robotics, vehicles, and machines etc. are often very complex and consist of many components interconnected by mechanical joints and force elements. The governing equations of such systems are often governed by index-3 differential algebraic equations (DAEs). Although mathematical modeling tools for multibody dynamics simulation have experienced a tremendous growth, most researches were based on the assumption that all parameters of multibody systems are deterministic. However, the realistic engineering multibody systems often operate under some degree of uncertainty which may be resulted from variability in their geometric or material parameters, or caused by the assembly process and manufacturing tolerances and/or wear, ageing and so on. Hence, the multibody dynamics models must account for these uncertainties for achieving the realistic predictions of the system responses.

Interval arithmetic has appeared several decades, but interval theory was not mainly concentrated until the appearance of Moore's work (Revol, Makino et al. 2005). Interval arithmetic can obtain the system response bounds quickly, because it is not a type of optimization algorithm which needs a large mount of iterations. However, interval arithmetic has its own drawback that is the calculation results may be overestimated too much caused by the wrapping effect. How to reduce the overestimation is the key for interval arithmetic. Many interval methods have been proposed to solve the static problems (Zingales and Elishakoff 2000; Chen, Lian et al. 2002; Gao 2006; Muhanna, Zhang et al. 2007; Wang, Elishakoff et al. 2009; Gao, Song et al. 2010). However, the interval methods for solving the dynamics problems which are expressed as differential equations including ODEs and DAEs are presented not much. The numerical methods for solving differential equations contain much iteration which aggravates the overestimation, so besides using the interval set theory, many other particular algorithms are introduced to reduce the overestimation. Interval Taylor series method (Nedialkov, Jackson et al. 1999; Alefeld and Mayer 2000; Jackson and Nedialkov 2002) and Taylor model method (Berz and Makino 1999) are the two important methods. The Taylor model uses higher order Taylor series to approximate the system responses and adds a remainder interval to guarantee the interval ranges contain all the possible results, which reduces the wrapping effect induced by the dependency of interval variables. Lin's VSPODE (Lin and Stadtherr 2007) combined the two methods to solve the ODEs with interval parameters, which made the interval results sharper. The mechanical dynamics problems are generally governed by DAEs, especially by the index-3 DAEs. The numerical solution of DAEs has a comparatively short history related to ODEs, still, numerically solving DAEs poses fundamental difficulties not encountered when solving ODEs (Negrut, Jay et al. 2009).

To reduce the overestimation of interval inclusion function, the Chebyshev inclusion function using the truncated Chebyshev series to calculate the bounds of function with interval parameters is proposed. The Chebyshev inclusion function can reduce the overestimation effectively, because it can be expressed as cosine functions which make the interval range sharper for non-monotonic functions. Utilizing the Chebyshev inclusion function on DAEs with interval parameters, the overestimation can be controlled effectively. For Mehler integral is an interpolation quadrature formula, the solutions at each interpolation point are needed, and the traditional HHT-I3 numerical method is used to produce the solutions at each interpolation point. At last, the interval arithmetic can be employed to calculate the bounds of solutions of DAEs based on the obtained Chebyshev inclusion function.

Modeling and Solving the Multibody Dynamics System

The constrained equations of the dynamics of multibody systems can be expressed as (Negrut, Jay et al. 2009)

$$\begin{aligned} &\dot{\mathbf{q}} = \mathbf{v} \\ &\mathbf{M}(\mathbf{q})\,\dot{\mathbf{v}} = \mathbf{Q}\left(t,\mathbf{q},\mathbf{v},\boldsymbol{\lambda},\mathbf{u}(t)\right) - \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}(\mathbf{q},t)\,\boldsymbol{\lambda}, \\ &\boldsymbol{\Phi}(\mathbf{q},t) = \mathbf{0} \end{aligned} \tag{1}$$

where $\mathbf{q} \in R^n$ are the generalized coordinates, $\mathbf{v} \in R^n$ are the generalized velocities, $\boldsymbol{\lambda} \in R^m$ are the Langrage multipliers, and $\mathbf{u}: R \to R^c$ represent time dependent external dynamics, e.g. control variables. The matrix **M(q)** is the generalized mass matrix, $\mathbf{Q}\left(t,\mathbf{q},\mathbf{v},\boldsymbol{\lambda},\mathbf{u}(t)\right)$ represents the vector of generalized applied forces, and $\boldsymbol{\Phi}(\mathbf{q},t)$ is the set of m holonomic constraints. The notation in bold denotes vector, while the notation in italic denotes scalar.

The classical numerical techniques for DAEs contain two classes: state-space methods and direct methods (Bauchau and Laulusa 2008). The major intrinsic drawback associated with state-space methods remains the expensive DAE to ODE reduction process that is further exacerbated in the context of implicit integration (Negrut, Jay et al. 2009). Direct methods discretize the constrained equations and transform the DAEs to algebraic equations at each integral step. Many direct methods have been proposed to solve the index-3 DAEs, such as the Newmark method (Newmark 1959), HHT-I3 (Negrut, Rampalli et al. 2007), and generalized α-method (Chung and Hulbert 1993) and so on. In this paper, we use the HHT-I3 method which would be described as follows.

Discretize the Eq. (1) with respect to time leads to the following equations (Negrut, Jay et al. 2009)

$$\begin{aligned} &\mathbf{q}_{n+1} = \mathbf{q}_n + h\dot{\mathbf{q}}_n + \frac{h^2}{2}\left[(1+2\beta)\mathbf{a}_n + 2\beta\mathbf{a}_{n+1}\right] \\ &\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + h\left[(1-\gamma)\mathbf{a}_n + \gamma\mathbf{a}_{n+1}\right] \\ &\frac{1}{1+\alpha}\left(\mathbf{M}(\mathbf{q})\mathbf{a}\right)_{n+1} - \left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} - \mathbf{Q}\right)_{n+1} - \frac{\alpha}{1+\alpha}\left(\boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} - \mathbf{Q}\right)_n = \mathbf{0} \\ &\frac{1}{\beta h^2}\boldsymbol{\Phi}\left(\mathbf{q}_{n+1},t_{n+1}\right) = \mathbf{0} \end{aligned}, \tag{2}$$

where h is the integration step-size, $\mathbf{a}_{n+1}$ is the approximation of $\ddot{\mathbf{q}}\left(t_n+(1+\alpha)h\right)$, and the initial value $\mathbf{a}_0$ can be set as $\mathbf{a}_0=\ddot{\mathbf{q}}_0$, subscript n denotes the nth integral step, and subscript $\mathbf{q}$ denotes the derivative of $\mathbf{q}$. α,β, and γ are the parameters of HHT-I3 method that confirm the conditions as follow:

$$\alpha\in[-1/3,0],\beta=\left(1-\alpha^2\right)/4,\gamma=1/2-\alpha. \tag{3}$$

The smaller value of α leads larger numerical dissipation for HHT-I3 method, but it makes the solution more stability. The last two equations of Eq. (2) are the nonlinear system of $\mathbf{w}_{n+1}=\begin{bmatrix}\mathbf{a}_{n+1} & \boldsymbol{\lambda}_{n+1}\end{bmatrix}^{\mathrm{T}}$, so the Newton method can be used to solve the system. The Newton method does not consider the uncertain parameters in the equations, and the method treating for uncertainties will be presented in following sections.

Interval Arithmetic

Let us define a real interval $[x]$ is a connected nonempty subset of real set R. It can be expressed as

$$[x]=\left[\underline{x},\bar{x}\right]=\left\{x\in R:\underline{x}\le x\le\bar{x}\right\}, \tag{4}$$

where $\underline{x}$ is the lower bound of interval $[x]$ which also can be noted as $inf([x])$; $\bar{x}$ is the upper bound of interval $[x]$ which also can be noted as $sup([x])$. The set of all intervals over R is denoted by IR where

$$IR=\left\{\left[\underline{x},\bar{x}\right]:\underline{x},\bar{x}\in R:\underline{x}\le\bar{x}\right\}. \tag{5}$$

Interval arithmetic operations are defined on the real set R such that the interval result closes all possible real result. Given the two real interval $[x]$and $[y]$,

$$[x]*[y]=\{x*y:x\in[x],y\in[y]\}\ \text{for}\ *\in\{+,-,\times,\div\}. \tag{6}$$

Consider a function f from R^n to R^m. The interval function $[f]$ from IR^n to IR^m is an inclusion function for f if

$$\forall[x]\in IR^n, f([x])\subset[f]([x]). \tag{7}$$

The direct calculation of an enclosure for a function using interval arithmetic will often lead to large overestimation. To make the result sharper, the higher order Taylor series expansion can be used. If the function f is $n+1$ times differentiable on the interval $[x]$, the nth-order Taylor inclusion function (Jaulin 2001) can be obtained as follows:

$$\left[f_{T_n}\right]([x])=f(x_c)+f'(x_c)[\Delta x]+\ldots+\frac{1}{n!}f^{(n)}(x_c)[\Delta x]^n+\frac{1}{(n+1)!}\left[f^{(n+1)}([x])\right][\Delta x]^{n+1}, \tag{8}$$

where x_c denotes the midpoint of $[x]$

$$x_c=mid([x])=\frac{1}{2}(\underline{x}+\bar{x}). \tag{9}$$

And $[\Delta x]$ is a symmetry interval of $[x]$, which is expressed by

$$[\Delta x]=\left[\frac{\underline{x}-\bar{x}}{2},\frac{\bar{x}-\underline{x}}{2}\right]. \tag{10}$$

In the above, the Eq. (8) calculates the rigorous enclosure for the function $f(x)$. The last term in the right hand side of Eq. (8) is usually neglected to obtain the approximate enclosure of $f(x)$ in engineering. Some specific interval function can be calculated through some special algorithms, e.g. the trigonometric function (Jaulin 2001) and so on..

Chebyshev Method for Multibody Dynamics System with Interval Parameters

Chebyshev inclusion function

If the function $f(x)$ is contained in $C[a, b]$, which means $f(x)$ is continuous on $[a, b]$, then it can be approximated as truncated Chebyshev series with degree n (Li, Wang et al. 2003), shown as follow

$$f(x) \approx p_n(x) = \frac{1}{2} f_0 + \sum_{i=1}^{n} f_i C_i(x), \tag{11}$$

where f_i are the constant coefficients, and $C_i(x)$ denotes the Chebyshev polynomial. The Chebyshev polynomial for $x \in [-1,1]$ of degree n is denoted by C_n and is defined by (Rivlin 1981)

$$C_n(x) = \cos n\theta, \tag{12}$$

where $\theta = \arccos(x) \in [0, \pi]$, n denotes the nonnegative integer. The Chebyshev polynomial on $[a, b]$ of degree n is also defined by Eq. (12), but here $\theta = \arccos\left(\frac{2x-(b+a)}{b-a}\right)$. For multi-dimension problem, the polynomials are the tensor product of each one-dimension polynomial. For example, the k dimensions Chebyshev polynomials of $x_i \in [-1,1]$, i=1,2,…, k can be expressed as

$$C_{n_1,n_2,\dots,n_k}(x_1,\dots,x_k) = \cos(n_1\theta_1)\dots\cos(n_k\theta_k), \tag{13}$$

where $\theta_i = \arccos(x_i)$. The corresponding multi-dimension function $f(\mathbf{x})$ can be approximated as

$$\mathbf{f}(\mathbf{x}) \approx \sum_{i_1=0}^{n} \dots \sum_{i_k=0}^{n} \left(\frac{1}{2}\right)^p \mathbf{f}_{i_1,\dots,i_k} C_{i_1,\dots,i_k}(\mathbf{x}), \tag{14}$$

where p denotes the total number of zero(s) to be occurred in the subscripts $i_1,\dots,i_k$, $C_{i_1,\dots,i_k}(\mathbf{x})$ is the k-dimensional Chebyshev polynomials given in Eq. (13), and $\mathbf{f}_{i_1,\dots,i_k}$ denotes the vector including the coefficients of Chebyshev polynomials which can be calculated by Eq. (15)

$$\mathbf{f}_{i_1,\dots,i_k} = \left(\frac{2}{\pi}\right)^k \int_0^\pi \dots \int_0^\pi \mathbf{f}(\cos\theta_1,\dots,\cos\theta_k)\cos i_1\theta_1 \dots \cos i_k\theta_k d\theta_1 \dots d\theta_k, \tag{15}$$

where k denotes the number of dimension, and subscript $i_1,\dots, i_k$ =0, 1,…, n. The numerical integral methods should be used to calculate the Eq. (15), and the Mehler integral method is suitable. The Mehler integral is a type of interpolation integral which can be expressed as

$$\mathbf{f}_{i_1,\dots,i_k} \approx \left(\frac{2}{m}\right)^k \sum_{j_1=1}^{m} \dots \sum_{j_k=1}^{m} \mathbf{f}(\cos\theta_{j_1},\dots,\cos\theta_{j_k})\cos i_1\theta_{j_1} \dots \cos i_k\theta_{j_k}, \tag{16}$$

where m denotes the number of interpolation points, θ_j denotes the interpolation points

$$\theta_j = \frac{2j-1}{m}\frac{\pi}{2}, j = 1,2,\dots,m. \tag{17}$$

Similar to Taylor inclusion function, we define the Chebyshev inclusion function of $f(x)$ which can be expressed as

$$[\mathbf{f}_{C_n}]([\mathbf{x}]) = \sum_{j_1=0}^{n} \dots \sum_{j_k=0}^{n} \left(\frac{1}{2}\right)^p \mathbf{f}_{j_1,\dots,j_k} \cos(j_1[\theta_1])\dots\cos(j_k[\theta_k]), \tag{18}$$

where $[\theta] = [0, \pi]$. Eq. (18) can be calculated through the algorithm of interval trigonometric function shown in section 3.

Chebyshev method for solving multibody systems containing interval parameters

From section 2, we know that the numerical method for solving the multibody dynamics system transform the DAEs to nonlinear equations at each integral step. When consider the uncertain parameters and uncertain external excitation are contained in the multibody system, such as the length tolerance of components inducing the mass and center of mass uncertain, the density uncertainty leading the mass and the moment of inertia uncertain, and the fluctuated driving force, the DAEs can be transformed to nonlinear equations containing uncertain parameters.

Since the DAEs are transformed to nonlinear equations at each iteration step, the nonlinear equations with interval parameters will be researched. Consider the q dimensions function group $\mathbf{F}=\left[f_1, f_2, \ldots, f_q\right]^{\mathrm{T}}$, where $f_i : x \in \mathbf{X} \subset R^q \rightarrow R, i=1,2,\ldots,q$. If the uncertain parameters which are expressed as interval parameters $\xi \subset [\mathbf{a},\mathbf{b}]^k$ exist in the nonlinear system, the nonlinear system can be described as

$$\mathbf{F}\left(\mathbf{X},\xi\right)=\mathbf{0}. \tag{19}$$

The solution set of Eq. (19) is a function with respect to uncertain parameters ξ, and its interval solution is $\left[\mathbf{X}([\xi])\right]$

$$\mathbf{X}(\xi)=\mathbf{F}^{-1}\left(\mathbf{Y},\xi\right)\Big|_{\mathbf{Y}=\mathbf{0}} \subset \left[\mathbf{X}\left([\xi]\right)\right]. \tag{20}$$

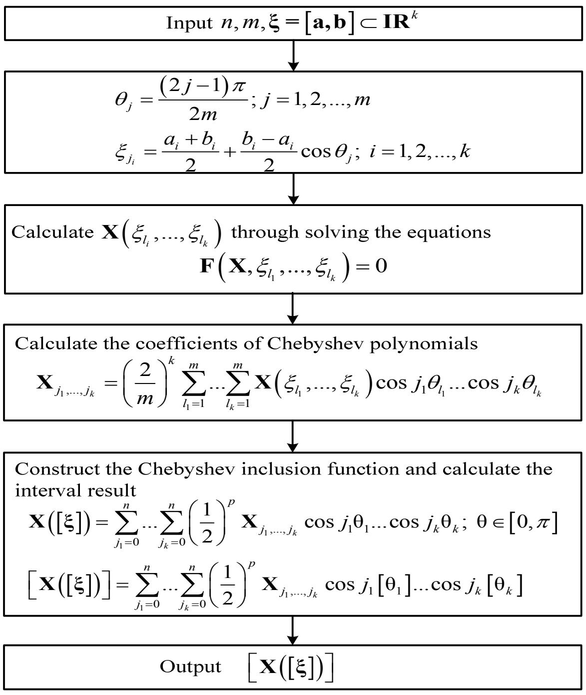
Considering the Chebyshev inclusion function Eq. (19), the interval solution $\left[\mathbf{X}([\xi])\right]$ can be calculated as

$$\left[\mathbf{X}([\xi])\right]=\sum_{j_1=0}^{n}\ldots\sum_{j_k=0}^{n}\left(\frac{1}{2}\right)^p \mathbf{X}_{j_1,\ldots,j_k}\cos\left(j_1[\theta_1]\right)\ldots\cos\left(j_k[\theta_k]\right), \tag{21}$$

where $\boldsymbol{\theta} \subset [0,\pi]^k$, and the coefficients vector $\mathbf{X}_{j_1,\ldots,j_k}$ can be obtained through Eq. (16)

$$\mathbf{X}_{j_1,\ldots,j_k}=\left(\frac{2}{m}\right)^k\sum_{l_1=1}^{m}\ldots\sum_{l_k=1}^{m}\mathbf{X}\left(\theta_{l_1},\ldots,\theta_{l_k}\right)\cos j_1\theta_{l_1}\ldots\cos j_k\theta_{l_k}, \tag{22}$$

where θ_l denotes the interpolation points expressed by Eq. (17), and $\mathbf{X}\left(\theta_{l_1},\ldots,\theta_{l_k}\right)$ denotes the solution of nonlinear system shown in Eq. (19) when the values of uncertain parameters are set as $\xi=\left[\cos\theta_{l_1},\ldots,\cos\theta_{l_k}\right]^{\mathrm{T}}$. The detail algorithm for solving multibody systems with uncertain parameters can be described as **Algorithm 1**.



Algorithm 1

From the calculation flow, we find that the algorithm solving the uncertain problem is similar to a type of sampling method, but its pre-processing and post-processing are particular. Thus, the proposed method can be used in black box problems even, but the accuracy and efficiency should be researched further.

Numerical Application

In this section, the numerical example which is slider crank mechanism containing interval parameters is presented. In the slider crank mechanism, the length of crank is firstly considered as an interval parameter, we hope to obtain the range of slider displacement in the whole calculation period. The schematic of slider crank is shown in Fig. 1, and the parameters are shown in Table 1.

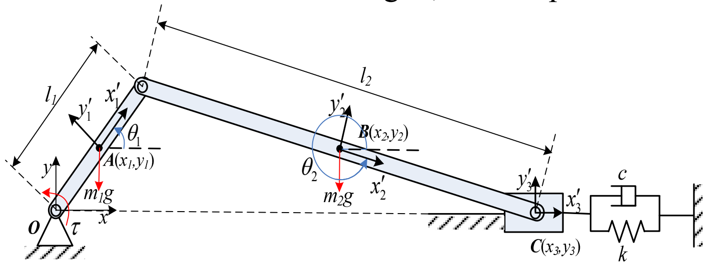


Figure. 1 The schematic of slider crank

Table 1. The parameters of slider crank

parameters	$l_1(m)$	$l_2(m)$	$m_1(kg)$	$m_2(kg)$	$m_3(kg)$	$c(N/(m/s))$	$k(N/m)$	τ (Nm)
value	0.15	0.56	0.37	0.77	0.45	1	5	-0.5

As shown in Fig. 1, point *A, B*, and *C* is the gravity center of crank, connecting rod, and slider respectively. θ_1 and θ_2 denotes the angle between the global coordinate and the local coordinate of crank and connecting rod respectively. The slider is connected with a spring damper, and the spring force is zero when the angle θ_1 and θ_2 equal to zero. l_1 and l_2 denotes the length of crank and connecting rod; m_1, m_2, and m_3 denotes the mass of crank, connecting rod, and slider respectively; c is the damp coefficient of spring damper, k is the stiffness of spring damper, and τ denotes the external torque applied on the crank. Choose the seven generalized coordinates which are $\mathbf{q}=\left[x_1, y_1, \theta_1, x_2, y_2, \theta_2, x_3\right]^T$, where the subscript 1, 2, and 3 denotes the crank, connecting rod, and slider, respectively. Suppose the length of crank l_1 containing uncertainty with 1% of its nominal value, noting it as

$$\hat{l}_1 = l_1\left(1+0.01\xi_1\right), \quad \xi_1 \in \left[-1,1\right]. \tag{23}$$

The system is solved for a period of 2*s* by using the Chebyshev method with 5th-order polynomials and the second-order Taylor method, respectively. To ensure the precise ranges of results, the scanning method (Buras, Jamin et al. 1996) is employed with symmetrical 30 sampling points. The results are shown in Fig. 2.

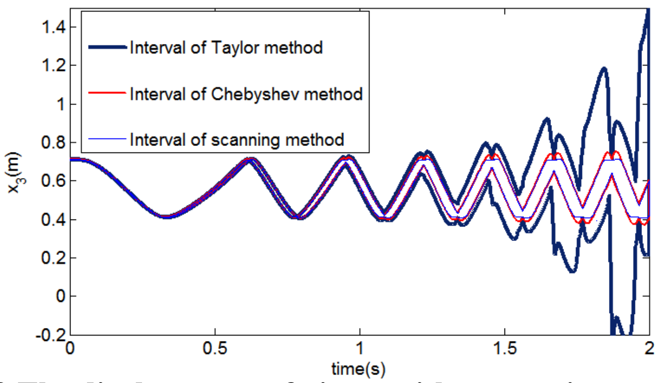


Figure. 2 The displacement of piston with uncertain crank length

The results obtained by Chebyshev method enclose the range of scanning method tightly in the initial period, compared with the Taylor inclusion function method.

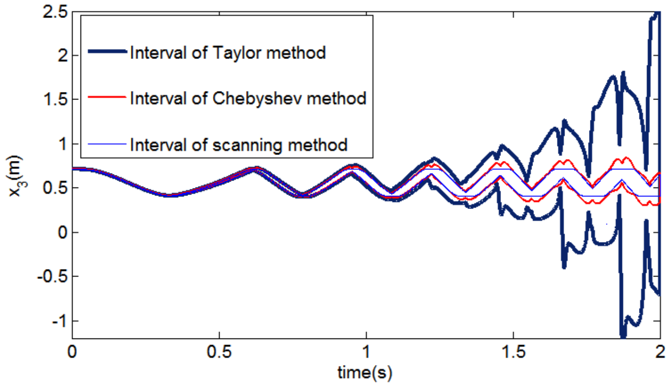


Figure. 3 The displacement of piston with uncertain crank length and torque

Secondly, we also consider the external torque τ under interval uncertainty with 1% of its nominal value, and the uncertain external torque is then expressed as

$$\hat{\tau} = \tau\left(1 + 0.01\xi_2\right), \quad \xi_2 \in [-1,1] \tag{24}$$

The initial conditions keep unchanged. Solve the system for a period of 2s using the Chebyshev method with the 5th-order polynomials and the second-order Taylor method. The results are shown as Fig. 3. For the computational time, the proposed Chebyshev method requires 422s, while the Taylor method and scanning method needs 1392s and 10584s, respectively.

Conclusions

A new interval numerical method using Chebyshev series to solve the multibody dynamics system with uncertainties is presented. Interval method is mainly used in the cases that only the bounds of uncertain parameters are known. To weaken the drawback of interval method, overestimated too much, the Chebyshev inclusion function which employs the truncated Chebyshev series to approximate the original function is proposed. The Chebyshev polynomials approximation theory is also used in solving the nonlinear system with interval parameters. To solve the multibody system dynamics problems containing uncertain parameters, the classical HHT-I3 method is used to transforms the DAEs to nonlinear systems at each integral step, so the proposed algorithm for solving nonlinear system with interval parameters can be ultilized. The numerical example of slider crank mechanism is presented, where the length of crank and torque forced on crank are considered as interval parameters. The numerical results show that the results of Chebshev method enclose the results of scanning method tighter than the Taylor method, also the Chebyshev method is higher efficient than both scanning method and Taylor method. The proposed method is similar to the sampling method which may even settle the black box problems.

References

Alefeld, G. and G. Mayer (2000). "Interval analysis: theory and applications." Journal of Computational and Applied Mathmatics **121**: 421-464.

Bauchau, O. A. and A. Laulusa (2008). "Review of Contemporary Approaches for Constraint Enforcement in Multibody Systems." Journal of Computational and Nonlinear Dynamics **3**(1): 011005.

Berz, M. and K. Makino (1999). "Efficient Control of the Dependency Problem Based on Taylor Model Methods." Reliable Computing **5**: 3-12.

Buras, A. J., M. Jamin, et al. (1996). "A 1996 Analysis of the CP. Violating Ratio ε′/ε." Physics Letters B **389**(4): 749-756.

Chen, S., H. Lian, et al. (2002). "Interval static displacement analysis for structures with interval parameters." International Journal for Numerical Methods in Engineering **53**(2): 393-407.

Chung, J. and G. Hulbert (1993). "A time integration algorithm for structural dynamics with improved numerical dissipation: The generalized-a method." Journal of Applied Mechanics **60**: 371-375.

Gao, W. (2006). "Interval Finite Element Analysis using Interval Factor Method." Computational Mechanics **39**(6): 709-717.

Gao, W., C. Song, et al. (2010). "Static response and reliability analysis of structural systems with random and interval properties." IOP Conference Series: Materials Science and Engineering **10**: 012200.

Jackson, K. R. and N. S. Nedialkov (2002). "Some recent advances in validated methods for IVPs for ODEs." Applied Mathematics and Computation **42**: 269-284.

Jaulin, L. (2001). Applied interval analysis: with examples in parameter and state estimation, robust control and robotics. New York, Springer.

Li, Q., N. Wang, et al. (2003). Numerical Analysis. Wuhan, Huazhong University of Science and Technology Press.

Lin, Y. and M. A. Stadtherr (2007). "Validated solutions of initial value problems for parametric ODEs." Applied Numerical Mathematics **57**(10): 1145-1162.

Muhanna, R. L., H. Zhang, et al. (2007). "Combined axial and bending stiffness in interval finite-element methods." Journal of Structural Engineering **133**(12): 1700-1709.

Nedialkov, N. S., K. R. Jackson, et al. (1999). "Validated solutions of initial value problems for ordinary differential equations." Applied Mathematics and Computation **105**: 21-68.

Negrut, D., L. O. Jay, et al. (2009). "A Discussion of Low-Order Numerical Integration Formulas for Rigid and Flexible Multibody Dynamics." Journal of Computational and Nonlinear Dynamics **4**(2): 021008.

Negrut, D., R. Rampalli, et al. (2007). "On an implementation of the Hilber–Hughes–Taylor method in the context of index 3 differential-algebraic equations of multibody dynamics." Journal of Computational and Nonlinear Dynamics **2**: 73-85.

Newmark, N. M. (1959). "A method of computation for structural dynamics." Journal of Engineering Mechanics Division: 67-94.

Revol, N., K. Makino, et al. (2005). "Taylor models and floating-point arithmetic: proof that arithmetic operations are validated in COSY☆." Journal of Logic and Algebraic Programming **64**(1): 135-154.

Rivlin, T. J. (1981). An Introduction to the Approximation of Functions. New York, Dover.

Wang, X., I. Elishakoff, et al. (2009). "Comparisons of Probabilistic and Two Nonprobabilistic Methods for Uncertain Imperfection Sensitivity of a Column on a Nonlinear Mixed Quadratic-Cubic Foundation." Journal of Applied Mechanics **76**(1): 011007.

Zingales, M. and I. Elishakoff (2000). "Anti-optimization versus probability in applied mechanics problem: vector uncertainty." Journal of Applied Mechanics **67**: 472-484.