Analytical solution for the transient response of functionally graded rectangular plates subjected to moving loads

Ta DuyHien[1], Hyuk-Chun Noh[1]*

[1]Department of Civil and Environmental Engineering,Sejong University, Seoul, South Korea.

*Corresponding author: cpebach@sejong.ac.kr

Abstract

An analysis scheme for the dynamic responses of functionally graded (FG) rectangular plates under moving loads is developed by using the third-order shear deformation plate theory (TSDT). It is assumed that material properties of the plate vary continuously in the thickness direction according to the power-law. The equations of motion are derived by using Hamilton's principle. Analytic solution of simply supported FG rectangular plates is presented by using state-space methods. The displacement and stresses are computed in the plates with various structural parameters, and the effects of these parameters, such as power-law exponent index, are discussed in detail. In addition, the effects of the moving load on the dynamic responses of the plates are investigated as well.

Keywords: Analytical solution, Dynamic response, functionally graded plate, moving loads.

1. Introduction

The study of dynamic response of plates subjected to moving loads is of interestand of importance as well in the engineering field, as some of the results can be applicable to understand the dynamic behavior of bridge. Most of the previous studies on the plates subjected to moving loads have used the classical plate theory (CPT) or the first order shear deformation theory (FSDT). Hiang*et al.*[5] used finite strip method to investigate the dynamic response of plate structure resting on an elastic foundation to moving loads. Lee *et al.*[7] investigated dynamic behaviors of single and two-span continuous composite plate structures subjected to multi-moving loads using finite element method. Vosoughi *et al.*[12] studies dynamic response of laminated rectangular plates on elastic foundation based on the higher order shear deformation theory and differential quadrature method. E.Ghafoori and M.Asghari[3] studied the dynamic response of laminated composite plates traversed by a moving mass or a moving force based on the first-order shear deformation theory using a finite element method. P.Malekzadeh *et al.*[8] used three-dimensional elasticity theory to investigate the dynamic response of cross-ply laminated thick plates subjected to moving load. Hao *et al.*[4]studied on nonlinear dynamic behavior of a simply supported functionally graded materials (FGMs) rectangular plate subjected to thermalmechanical loads. Qian *et al.*[9] studied a static and dynamic of rectangular functionally graded plate based on a higher-order shear and normal deformable plate theory by using a meshless local Petrov–Galerkin method. Akbarzadeh*et et al.*[1] studies the dynamic response of a simply supported functionally graded rectangular plate subjected to thermomechanical loading by using the hybrid Fourier-Laplace transform method based on both the first-order and third-order shear deformation theories. Sun *et al.*[11] investigated the wave propagation and dynamic response of the rectangular FGM plates with completed clamped supports to consider the effects of transverse shear deformation and rotary inertia. However, the research works concerning the dynamic response of FG plates subjected to moving loads are still limited.

In the this paper, exact solutions for the transient response of FG rectangular plate are developed using the third-order shear deformation plate theory (TSDT) proposed by R.P. Shimpi[10]. The FG rectangular plate, with simply supported boundary conditions, is subjected to a concentrated moving load at the upper surface of the plate. Equations of motion are solved by using state-space methods and the effects of different parameters on the response of the plate are studied. The results are compared with finite element solutions for validation.

2. Equations of motion

Let us consider a functionally graded plate in Figure.1:

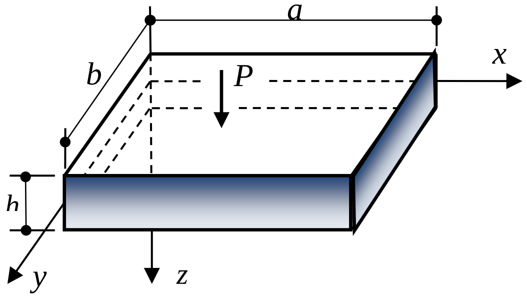

Figure 1.Model of FG Plate

We assume that the gradation of material properties along the plate thickness is represented by the profile for volume fraction variation:

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + E_m$$
$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + \rho_m \tag{1}$$

where E, ρ denote generic properties of elastic modulus and mass density, E_c, ρ_c and E_m, ρ_m denote the properties on the top and bottom surfaces, respectively, and is a parameter that dictates the material variation profile through the thickness. Poisson's ratios are assumed to be uniform.

The third-order shear deformation theory used in the present study is based on the following representation of the displacement field across the plate thickness as in Shimpi [10]:

$$U(x,y,z,t) = u(x,y,z,t) - z\frac{\partial w_b}{\partial x} + \left[\frac{1}{4}z - \frac{5}{3}z\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x}$$
$$V(x,y,z,t) = v(x,y,z,t) - z\frac{\partial w_b}{\partial y} + \left[\frac{1}{4}z - \frac{5}{3}z\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial y} \tag{2}$$
$$W(x,y,z,t) = w_b(x,y,t) + w_s(x,y,t)$$

where *(U,V,W)* are the displacement components of a point *(x,y,z)* in the plate, (u, v, w_b, w_s) are the displacements of a point on the mid-plane at time t, respectively.

The strains are computed from displacement fields in Eq. (2), and can be used in constructing the strain energy and kinetic energy expressions. Hamilton's principle is used herein to derive the equations of motion; the equations of motion of plate are obtained as:

$$\delta u: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\ddot{u} - I_1\frac{\partial \ddot{w}_b}{\partial x} - J_1\frac{\partial \ddot{w}_s}{\partial x}$$
$$\delta v: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0\ddot{v} - I_1\frac{\partial \ddot{w}_b}{\partial y} - J_1\frac{\partial \ddot{w}_s}{\partial y}$$
$$\delta w_b: \frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_y^b}{\partial y^2} + P = I_0(\ddot{w}_b + \ddot{w}_s) + I_1\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) - I_2\nabla^2\ddot{w}_b - J_2\nabla^2\ddot{w}_s \tag{3}$$
$$\delta w_s: \frac{\partial^2 M_x^s}{\partial x^2} + 2\frac{\partial^2 M_{xy}^s}{\partial x \partial y} + \frac{\partial^2 M_y^s}{\partial y^2} + \frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + P = I_0(\ddot{w}_b + \ddot{w}_s) + J_1\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) - \hat{J}_2\nabla^2\ddot{w}_b$$

where P is the moving load and $\left(N_i, Q_i, M_i^{b,s}\right)$ are the stress resultants and the inertias $\left(I_i, J_i, \hat{J}\right)$ are defined by:

$$\left(N_x, N_y, N_{xy}\right) = \int_{-h/2}^{h/2} \left(\sigma_x, \sigma_y, \sigma_{xy}\right) dz$$
$$\left(M_x^b, M_y^b, M_{xy}^b\right) = \int_{-h/2}^{h/2} z\left(\sigma_x, \sigma_y, \sigma_{xy}\right) dz$$
$$\left(M_x^s, M_y^s, M_{xy}^s\right) = \int_{-h/2}^{h/2} \hat{f}\left(\sigma_x, \sigma_y, \sigma_{xy}\right) dz$$
$$\left(Q_x, Q_y\right) = \int_{-h/2}^{h/2} \hat{g}\left(\sigma_{xz}, \sigma_{yz}\right) dz$$
$$I_i = \int_{-h/2}^{h/2} \rho(z) z^i dz, \quad (i = 0,1,2,3,4,6)$$
$$J_i = \int_{-h/2}^{h/2} \rho(z) z^i \hat{f} dz, \quad (i = 1,2)$$
$$\hat{J}_2 = \int_{-h/2}^{h/2} \rho(z) z^2 \hat{f}^2 dz$$

The moving loadis given as

$$P = P(t)\delta\left(x - x_{mov}(t)\right)\delta\left(y - y_{mov}(t)\right)$$

where $P(t)$ is the magnitude of the moving load; $x_{mov}(t)$ and $y_{mov}(t)$ are the coordinates of the location of the load. In this paper, it is assumed that $P = P_0$, $x_{mov}(t) = V_0 t$ and $y_{mov}(t) = b/2$.

3 Solution procedures

The state-space approach has been used generally in the area of control theory to determine the responses of given systems. The state-space representations [6] of the dynamic systems will be used to analyze the transient response of simply supported FG rectangular plates with side dimensions a and b. The Navier approach is used to derive the closed-form solutions of equations of motion. The sinusoidal function is chosen to satisfy all the boundary conditions as follows:

$$\begin{aligned}
u(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_{mn}(t)\cos\alpha x \sin\beta y \\
v(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{mn}(t)\sin\alpha x \cos\beta y \\
w_b(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{bmn}(t)\sin\alpha x \sin\beta y \\
w_s(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{smn}(t)\sin\alpha x \sin\beta y
\end{aligned} \tag{4}$$

where : $\alpha = \dfrac{m\pi}{a}, \beta = \dfrac{n\pi}{b}$.

Using (4), the expression in (3) can be written as follows:

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{12} & s_{22} & s_{23} & s_{24} \\ s_{13} & s_{23} & s_{33} & s_{34} \\ s_{14} & s_{24} & s_{34} & s_{44} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix} + \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{43} & m_{44} \end{bmatrix} \begin{Bmatrix} \ddot{U}_{mn} \\ \ddot{V}_{mn} \\ \ddot{W}_{bmn} \\ \ddot{W}_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{mn} \\ F_{mn} \end{Bmatrix} \tag{5}$$

Here, we will use the following notations:

$$\boldsymbol{M} = \left\{m_{ij}\right\}, \ \boldsymbol{S} = \left\{s_{ij}\right\}$$
$$\boldsymbol{F}^* = \left\{0 \quad 0 \quad F_{mn} \quad F_{mn}\right\}^T$$

where,

$$F_{mn} = \frac{4P_0}{ab}\sin\left(\alpha V_0 t\right)\sin\left(\frac{\beta b}{2}\right)$$

For solving (5) by using the state space methods, (5) is needed to be rewritten as:

$$\dot{\boldsymbol{Z}} = \boldsymbol{AZ} + \boldsymbol{b} \qquad (6)$$

where

$$\boldsymbol{Z} = \left\{U_{mn} \quad V_{mn} \quad W_{bmn} \quad W_{smn} \quad \dot{U}_{mn} \quad \dot{V}_{mn} \quad \dot{W}_{bmn} \quad \dot{W}_{smn}\right\}^T$$

$$\boldsymbol{b} = \left\{0 \quad 0 \quad 0 \quad 0 \quad b_1 \quad b_2 \quad b_3 \quad b_4\right\}^T$$

and $\boldsymbol{b_i}$ are the term of the column matrix

$$\left\{b_1 \quad b_2 \quad b_3 \quad b_4\right\}^T = \boldsymbol{M}^{-1}\boldsymbol{F}^*$$

and block matrix A is

$$\boldsymbol{A} = \begin{bmatrix} 0 & I \\ -\boldsymbol{M}^{-1}\boldsymbol{S} & 0 \end{bmatrix}$$

The solution of (6) is obtained as

$$\dot{\boldsymbol{Z}}\left(t\right) = e^{\boldsymbol{A}\left(t-t_0\right)}\boldsymbol{Z}\left(t_0\right) + \int_{t_0}^{t} e^{\boldsymbol{A}\left(t-\tau\right)}\boldsymbol{b}\left(\tau\right)d\tau \qquad (7)$$

where t_0 is the initial time, $\boldsymbol{Z}\left(t_0\right)$ is the initial response, and the exponential matrix $e^{\boldsymbol{A}\left(t-t_0\right)}$ can be expressed in terms of the matrix of eigenvectors and eigenvalues λ_i associated with the matrix $\boldsymbol{A}$.

4 Numerical examples

In order to investigate the legitimacy of the proposed method, an Al/Al2O3 plate composed of aluminum (as metal) and alumina (as ceramic) is considered. The elastic moduli are chosen to be the same as given in [2]: $E_m = 70\ GPa$, $\rho_m = 2702 kg/m^3$, $E_c = 380 GPa$, $\rho_c = 3800 kg/m^3$. The Poisson's ratio of the plate is assumed to be constant through the thickness and is equal to 0.3. Consider a simply supported square FG plate with a side-length $a = 0.3m$ and magnitude of load $P_0 = 2\times 10^3 N$.

To validate the presented approach, the results are compared with those of finite element solution by using ABAQUS software. The time histories of central deflection and normal stress at top*(z=-h/2)* as functions of time for simply supported square FG plate with $p = 2$, $h/a = 0.05$ and $v = 15m/s$, are plotted and compared in Figures 2 and 3. The results of this study are in good agreement with the finite element solutions.

Figure4 contains plots of center transverse deflection as functions of time with parameter $h/a = 0.05$ and $v = 40m/s$ for the four cases with different power law index p=0,1,2,3, respectively. Figure4 shows that the oscillation of the plate deflection increases if power law index p increases, which makes the flexural rigidity high.

In order to investigate the effect of velocity of moving loads on the plate responses, we consider square FG plate with parameter $a = 0.3m$, $h/a = 0.05$ and $p = 3$ for three cases of different velocities v=1,25,50m/s, respectively. The effect of velocity of moving loads can be seen from Figure 5; the effect of velocity of moving loads is not so huge for the case of displacement. However, the effect of moving load velocity is significant from the viewpoint of vibration of the plate.

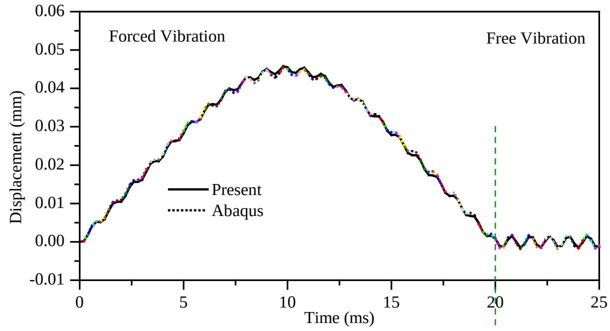

Figure 2. The time variations of the deflection at central plate

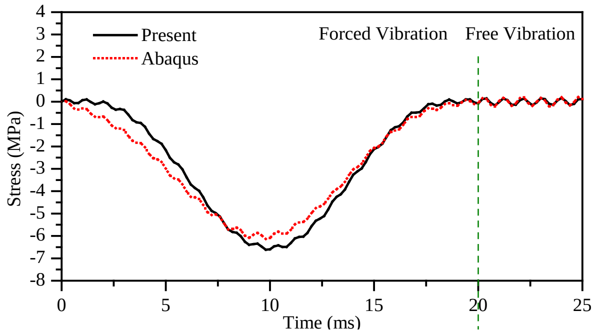

Figure 3. The time variations of the normal stress at top (*z*=-*h*/2) of central plate

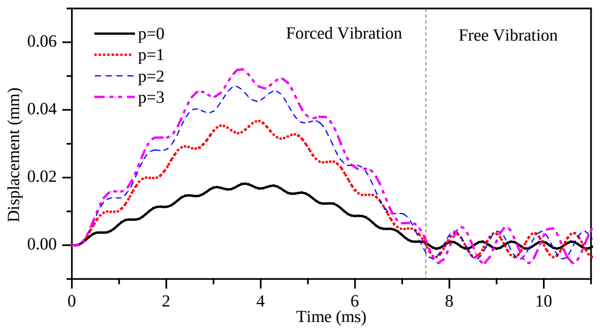

Figure 4. The time variations of the deflection at central plate with various power law *p*

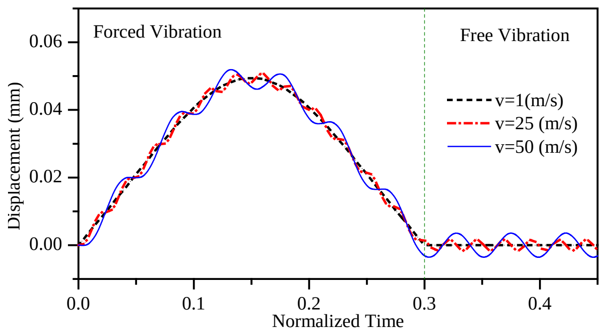

Figure 5. The time variations of the deflection at central plate with various velocity *v*

Conclusions

In this article, analytical solutions of dynamic responses are obtained for FG plates employing a third-order plate theory base on state-space approach. The dynamic responses are considered for both forced vibration and free vibration. Comparison with finite element solution shows that the results of present approach are acceptable. The volume fraction power law along the plate thickness has great influence on the dynamic behavior of FG plate, and the deflections of plate can be controlled by choosing proper values of p.The results also confirm that the effect of velocity of moving loads is negligible.

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